Math 180, Exam 1, Spring 2011
Problem 1 Solution

1. Evaluate the following limits, or show that they do not exist

(a) $\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 4}$

(b) $\lim_{x \to 3} \frac{|x^2 - 9|}{x^2 + 9}$

(c) $\lim_{x \to 2} \frac{x - 2}{\sqrt{x} - \sqrt{2}}$

Solution:

(a) Upon substituting $x = 2$ we find that

$$\frac{x^2 + x - 6}{x^2 - 4} = \frac{2^2 + 2 - 6}{2^2 - 4} = \frac{0}{0}$$

which is indeterminate. To resolve the indeterminacy we factor the numerator and denominator, cancel terms, and evaluate the resulting limit.

$$\begin{aligned} \lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 4} &= \lim_{x \to 2} \frac{(x+3)(x-2)}{(x+2)(x-2)} \\ &= \lim_{x \to 2} \frac{x+3}{x+2} \\ &= \frac{2+3}{2+2} \\ &= \boxed{\frac{5}{4}} \end{aligned}$$

We were able to substitute $x = 2$ after canceling the $x - 2$ terms because the function $\frac{x+3}{x+2}$ is continuous at $x = 2$.

(b) Upon substituting $x = 3$ we find that:

$$\lim_{x \to 3} \frac{|x^2 - 9|}{x^2 + 9} = \frac{|3^2 - 9|}{3^2 + 9} = \frac{0}{18} = \boxed{0}$$

The substitution method works here because the function $\frac{|x^2-9|}{x^2+9}$ is continuous everywhere.

(c) Upon substituting $x = 2$ we find that

$$\frac{x-2}{\sqrt{x}-\sqrt{2}} = \frac{2-2}{\sqrt{2}-\sqrt{2}} = \frac{0}{0}$$

which is indeterminate. To resolve the indeterminacy we multiply the numerator and denominator by $\sqrt{x}+\sqrt{2}$, cancel terms, and evaluate the resulting limit.

$$\begin{aligned}
\lim_{x\to 2} \frac{x-2}{\sqrt{x}-\sqrt{2}} &= \lim_{x\to 2} \frac{x-2}{\sqrt{x}-\sqrt{2}} \cdot \frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}} \\
&= \lim_{x\to 2} \frac{(x-2)(\sqrt{x}+\sqrt{2})}{x-2} \\
&= \lim_{x\to 2} \left(\sqrt{x}+\sqrt{2}\right) \\
&= \sqrt{2}+\sqrt{2} \\
&= \boxed{2\sqrt{2}}
\end{aligned}$$

We were able to substitute $x = 2$ after canceling the $x - 2$ terms because the function $\sqrt{x}+\sqrt{2}$ is continuous at $x = 2$.

Math 180, Exam 1, Spring 2011
Problem 2 Solution

2. Consider the equation $x^3 + x + 1 = 0$.

   (a) Use the Intermediate Value Theorem to show that it has a solution in the interval $[-2, 0]$.

   (b) Use the Bisection Method to find an interval of length $\frac{1}{2}$ that contains a solution.

Solution: Let $f(x) = x^3 + x + 1$.

(a) Since $f(x)$ is a polynomial, we know that it is continuous everywhere. Furthermore, $f(0) = 1$ and $f(-2) = -9$ have opposite signs. Therefore, the Intermediate Value Theorem guarantees the existence of a zero of $f(x)$ on the interval $[-2, 0]$.

(b) The midpoint of $[-2, 0]$ is $x = -1$. Since $f(-1) = -1$ and $f(0) = 1$ have opposite signs, we know that a zero of $f(x)$ exists in the interval $[-1, 0]$.

The midpoint of $[-1, 0]$ is $x = -\frac{1}{2}$. Since $f(-\frac{1}{2}) = \frac{3}{8}$ and $f(-1) = -1$ have opposite signs, we know that a zero of $f(x)$ exists in the interval $\boxed{[-1, -\frac{1}{2}]}$. This interval has a length of $\frac{1}{2}$ so we're done.

Math 180, Exam 1, Spring 2011
Problem 3 Solution

3. Let $f(x) = x^2 - 2x - 2$.

(a) Use the definition of the derivative as a limit of difference quotients to compute $f'(3)$.

(b) Find an equation of the tangent line to the graph of f at the point $(3, 1)$.

Solution:

(a) Using the limit definition of the derivative, we can compute $f'(3)$ using either of the following formulas:

$$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}$$

Using the first formula, we get:

$$\begin{aligned} f'(3) &= \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} \\ &= \lim_{x \to 3} \frac{(x^2 - 2x - 2) - (3^2 - 2(3) - 2)}{x - 3} \\ &= \lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3} \\ &= \lim_{x \to 3} \frac{(x + 1)(x - 3)}{x - 3} \\ &= \lim_{x \to 3} (x + 1) \\ &= 3 + 1 \\ &= \boxed{4} \end{aligned}$$

(b) Using the fact that $f'(3) = 4$ is the slope of the tangent line at the point $(3, 1)$, an equation for the tangent line in point-slope form is:

$$\boxed{y - 1 = 4(x - 3)}$$

Math 180, Exam 1, Spring 2011
Problem 4 Solution

4. Let $f(x) = \frac{1}{1-x} + 3$.

(a) Find the average rate of change of the function between $x = -0.6$ and $x = -0.4$.

(b) Find the instantaneous rate of change at $x = -0.5$.

Solution:

(a) The average rate of change of $f(x)$ on the interval $[-0.6, -0.4]$ is:

$$\begin{aligned}\text{average ROC} &= \frac{f(-0.4) - f(-0.6)}{-0.4 - (-0.6)} \\ &= \frac{\left(\frac{1}{1-(-0.4)} + 3\right) - \left(\frac{1}{1-(-0.6)} + 3\right)}{-0.4 - (-0.6)} \\ &= \frac{\frac{1}{1.4} - \frac{1}{1.6}}{0.2} \\ &= \boxed{\frac{25}{56}}\end{aligned}$$

(b) The instantaneous rate of change at $x = -0.5$ is $f'(-0.5)$. The derivative $f'(x)$ is found using the Chain Rule.

$$\begin{aligned}f'(x) &= \left(\frac{1}{1-x} + 3\right)' \\ &= -(1-x)^{-2} \cdot (1-x)' + 3' \\ &= \frac{1}{(1-x)^2}\end{aligned}$$

At $x = -0.5$, we have:

$$f'(-0.5) = \frac{1}{(1-(-0.5))^2} = \boxed{\frac{4}{9}}$$

Math 180, Exam 1, Spring 2011
Problem 5 Solution

5. Find the derivatives of the following functions using the basic rules. Leave your answers in an unsimplified form so that it is clear what method you used.

(a) $f(x) = \sin(x^3)$

(b) $f(x) = x^2 \cdot \arctan(3x)$

(c) $f(x) = \dfrac{1 - \cos x}{x^2 + 1}$

(d) $f(x) = x^3 e^{-x}$.

Solution:

(a) Use the Chain Rule.

$$\begin{aligned} f'(x) &= \left[\sin\left(x^3\right)\right]' \\ &= \cos\left(x^3\right) \cdot \left(x^3\right)' \\ &= \boxed{\cos\left(x^3\right) \cdot 3x^2} \end{aligned}$$

(b) Use the Product and Chain Rules.

$$\begin{aligned} f'(x) &= \left[x^2 \cdot \arctan(3x)\right]' \\ &= x^2 \cdot [\arctan(3x)]' + \left(x^2\right)' \cdot \arctan(3x) \\ &= x^2 \cdot \frac{1}{1 + (3x)^2} \cdot (3x)' + 2x \cdot \arctan(3x) \\ &= \boxed{x^2 \cdot \frac{1}{1 + (3x)^2} \cdot 3 + 2x \cdot \arctan(3x)} \end{aligned}$$

(c) Use the Quotient Rule.

$$\begin{aligned} f'(x) &= \left(\frac{1 - \cos x}{x^2 + 1}\right)' \\ &= \frac{(x^2 + 1)(1 - \cos x)' - (1 - \cos x)(x^2 + 1)'}{(x^2 + 1)^2} \\ &= \boxed{\frac{(x^2 + 1)(\sin x) - (1 - \cos x)(2x)}{(x^2 + 1)^2}} \end{aligned}$$

(d) Use the Product and Chain Rules.

$$
\begin{aligned}
f'(x) &= \left(x^3 e^{-x}\right)' \\
&= x^3 \left(e^{-x}\right)' + e^{-x} \left(x^3\right)' \\
&= \boxed{x^3 \left(-e^{-x}\right) + e^{-x} \left(3x^2\right)}
\end{aligned}
$$

Math 180, Exam 1, Spring 2011
Problem 6 Solution

6. Find the value/s of c for which the function

$$f(x) = \begin{cases} x^2 + 3 & \text{if } x < 2 \\ cx - 1 & \text{if } x \geq 2 \end{cases}$$

is continuous at $x = 2$. Justify your answers.

Solution: $f(x)$ will be continuous at $x = 2$ if

$$\lim_{x \to 2} f(x) = f(2)$$

To determine the limit, we must consider the one-sided limits as $x \to 2$. The limit as $x \to 2^-$ is

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} \left(x^2 + 3\right) \\ &= 2^2 + 3 \\ &= 7 \end{aligned}$$

The limit as $x \to 2^+$ is

$$\begin{aligned} \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} (cx - 1) \\ &= c(2) - 1 \\ &= 2c - 1 \end{aligned}$$

In order for the limit to exist, the one-sided limits must be the same. So we must have:

$$\begin{aligned} \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^-} f(x) \\ 2c - 1 &= 7 \\ c &= 4 \end{aligned}$$

Thus, when $c = 4$ the one-sided limits are the same and both are equal to 7. Furthermore, when $c = 4$ we know that $f(2) = 4(2) - 1 = 7$, so the function is continuous at $x = 2$ when $\boxed{c = 4}$.