Math 180, Exam 1, Spring 2012
Problem 1 Solution

1. Let c be a real number. Given

$$f(x) = \begin{cases} (x-1)^2 & x \leq 3 \\ 6 - cx & x > 3 \end{cases}$$

answer the following questions:

(a) Compute the left hand limit, $\lim_{x \to 3^-} f(x)$.

(b) Compute the right hand limit, $\lim_{x \to 3^+} f(x)$.

(c) Compute the value of c for which the two-sided limit $\lim_{x \to 3} f(x)$ exists.

(d) Is $f(x)$ continuous at $x = 2$? Explain.

Solution:

(a) Since x approaches 3 from the left we know that $x < 3$ so we use $f(x) = (x-1)^2$. The limit is then

$$\lim_{x \to 3^-} (x-1)^2 = (3-1)^2 = \boxed{4}.$$

(b) Since x approaches 3 from the right we know that $x > 3$ so we use $f(x) = 6 - cx$. The limit is then

$$\lim_{x \to 3^+} (6 - cx) = 6 - c(3) = \boxed{6 - 3c}.$$

(c) In order for the two-sided limit to exist, the one-sided limits have to be the same. That is,

$$\begin{aligned} \lim_{x \to 3^+} f(x) &= \lim_{x \to 3^-} f(x) \\ 6 - 3c &= 4 \\ -3c &= -2 \\ \Aboxed{c &= \frac{2}{3}} \end{aligned}$$

(d) Since $x < 3$ we use $f(x) = (x-1)^2$. This function is a polynomial and is continuous everywhere in its domain. Therefore, it is continuous at $x = 2$.

Math 180, Exam 1, Spring 2012
Problem 2 Solution

2. Find each limit or explain why it does not exist.

(a) $\displaystyle\lim_{x\to 0} \frac{3x^2+4}{x^2+6x+1}$.

(b) $\displaystyle\lim_{x\to -2} \frac{x^2+3x+2}{2x(x+2)}$.

(c) $\displaystyle\lim_{x\to \infty} \frac{x^{10}-2x^3+5}{x^2+1}$.

Solution:

(a) The given function is rational and we know that it is continuous everywhere except when the denominator is 0. Since $x = 0$ is not a root of the denominator so we know that the function is continuous there and we may evaluate the limit using substitution.

$$\lim_{x\to 0} \frac{3x^2+4}{x^2+6x+1} = \frac{3(0)^2+4}{0^2+6(0)+1} = \boxed{4}$$

(b) Upon substituting $x = -2$ into the given function we find that both the numerator and denominator tend to 0. We resolve this indeterminacy by factoring.

$$\lim_{x\to -2} \frac{x^2+3x+2}{2x(x+2)} = \lim_{x\to -2} \frac{(x+1)(x+2)}{2x(x+2)},$$
$$\lim_{x\to -2} \frac{x^2+3x+2}{2x(x+2)} = \lim_{x\to -2} \frac{x+1}{2x},$$
$$\lim_{x\to -2} \frac{x^2+3x+2}{2x(x+2)} = \frac{-2+1}{2(-2)},$$
$$\lim_{x\to -2} \frac{x^2+3x+2}{2x(x+2)} = \boxed{\frac{1}{4}}.$$

(c) The given function is rational and the degree of the numerator is greater than the degree of the denominator. Furthermore, the coefficients of the leading terms in the numerator and denominator are positive. Therefore, we know that the limit as $x \to \infty$ will be ∞. That is,

$$\lim_{x\to \infty} \frac{x^{10}-2x^3+5}{x^2+1} = \boxed{\infty}.$$

Math 180, Exam 1, Spring 2012
Problem 3 Solution

3. Compute the derivative of each function below. Leave your answer in an unsimplified form so it is clear what method you used.

(a) $x^2 \sin(x)$

(b) $\dfrac{e^x - e^{-x}}{1+x}$

(c) $\tan(\sqrt{x})$.

Solution:

(a) Use the Product Rule.

$$\frac{d}{dx}x^2\sin(x) = x^2\frac{d}{dx}\sin(x) + \sin(x)\frac{d}{dx}x^2,$$
$$\frac{d}{dx}x^2\sin(x) = \boxed{x^2\cos(x) + 2x\sin(x)}.$$

(b) Use the Quotient Rule.

$$\frac{d}{dx}\frac{e^x - e^{-x}}{1+x} = \frac{(1+x)\frac{d}{dx}(e^x - e^{-x}) - (e^x - e^{-x})\frac{d}{dx}(1+x)}{(1+x)^2},$$
$$\frac{d}{dx}\frac{e^x - e^{-x}}{1+x} = \boxed{\frac{(1+x)(e^x + e^{-x}) - (e^x - e^{-x})}{(1+x)^2}}.$$

(c) Use the Chain Rule.

$$\frac{d}{dx}\tan(\sqrt{x}) = \sec^2(\sqrt{x})\frac{d}{dx}\sqrt{x},$$
$$\frac{d}{dx}\tan(\sqrt{x}) = \boxed{\sec^2(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}}.$$

Math 180, Exam 1, Spring 2012
Problem 4 Solution

4. Use the Intermediate Value Theorem to show that the equation $x^3 = 3x+1$ has a solution.

Solution: We begin by rearranging the equation so that the right hand side is zero. That is,

$$x^3 - 3x - 1 = 0.$$

Let $f(x) = x^3 - 3x - 1$. We know that $f(x)$ is continuous everywhere because it is a polynomial. Furthermore, we know that

$$f(0) = 0^3 - 3(0) - 1 = -1,$$
$$f(2) = 2^3 - 3(2) - 1 = 1$$

have opposite signs. Therefore, the Intermediate Value Theorem guarantees the existence of at least one c such that $f(c) = 0$ on the interval $(0, 2)$.

Note: The interval $(0, 2)$ is not unique. For example, we know that $f(1) = -3 < 0$ and $f(2) = 1 > 0$ so the IVT guarantees at least one solution in the interval $(1, 2)$.

Math 180, Exam 1, Spring 2012
Problem 5 Solution

5. Given the function $f(x) = x^2 + 1$ answer the following:

(a) Use the limit definition of the derivative to compute $f'(1)$.

[**Note**: Using anything but the limit definition to compute $f'(1)$ will result in little to no credit.]

(b) Give the equation of the tangent line to the graph of f at the point $(1, 2)$.

Solution:

(a) We use the following definition to compute $f'(1)$:

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

Using the fact that $f(x) = x^2 + 1$ and $f(1) = 2$ we get

$$\begin{aligned}
f'(1) &= \lim_{x \to 1} \frac{x^2 + 1 - 2}{x - 1}, \\
f'(1) &= \lim_{x \to 1} \frac{x^2 - 1}{x - 1}, \\
f'(1) &= \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1}, \\
f'(1) &= \lim_{x \to 1} (x + 1), \\
f'(1) &= 1 + 1, \\
&\boxed{f'(1) = 2}
\end{aligned}$$

(b) The derivative $f'(1) = 2$ represents the slope of the tangent line at the point $(1, 2)$. Thus, the equation of the tangent line in point-slope form is:

$$\boxed{y - 2 = 2(x - 1)}$$