Math 180, Exam 1, Spring 2013
Problem 1 Solution

1. Find the value of constant c for which the function given by

$$f(x) = \begin{cases} cx+5, & x \geq 1 \\ x^2+x-3c, & x < 1 \end{cases}$$

is continuous at all points on the real line.

Solution: First we note that $cx+5$ and x^2+x-3c are polynomials and are continuous on the intervals $x > 1$ and $x < 1$, respectively. We must determine the constant c so that $f(x)$ is continuous at $x = 1$. Recall that for continuity at $x = 1$ we need $\lim_{x \to 1} f(x)$ to exist.

The one-sided limits of $f(x)$ at $x = 1$ are:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (cx+5) = c+5$$
$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2+x-3c) = 2-3c$$

In order for $\lim_{x \to 1} f(x)$ to exist we need the one-sided limits to be the same. That is, we need:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$$
$$c+5 = 2-3c$$
$$4c = -3$$

Answer $\boxed{c = -\frac{3}{4}}$

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Problem 2 Solution

2. Find an equation for the tangent line to the graph of the function $f(x) = \sin(x)$ at the point $x = \pi/4$.

Solution: The derivative of $f(x)$ at $x = \frac{\pi}{4}$ is the slope of the tangent line. The derivative of f is $f'(x) = \cos(x)$. At $t = \frac{\pi}{4}$ we have

$$f'(\tfrac{\pi}{4}) = \cos(\tfrac{\pi}{4}) = \tfrac{\sqrt{2}}{2}$$

Thus, the slope of the tangent line is $m_{\text{tan}} = \frac{\sqrt{2}}{2}$. The y-coordinate of the point on the tangent line is obtained by evaluating $f(x)$ at $x = \frac{\pi}{4}$.

$$f'(\tfrac{\pi}{4}) = \sin(\tfrac{\pi}{4}) = \tfrac{\sqrt{2}}{2}$$

Therefore, the point on the tangent line is $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$ and the equation for the tangent line in point-slope form is:

ANSWER $$\boxed{y - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)}$$

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Problem 3 Solution

3. Find the derivative of f if

(a) $f(x) = \sqrt{\cot(e^x)}$

(b) $f(t) = \dfrac{t + \tan(t)}{\sqrt{t} + 1}$

Solution:

(a) The derivative is obtained using the Chain Rule and the fact that

$$\frac{d}{dx}\cot(x) = -\csc^2(x) \quad \text{and} \quad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

We obtain

$$f'(x) = \frac{1}{2\sqrt{\cot(e^x)}} \cdot \frac{d}{dx}\cot(e^x)$$

$$f'(x) = \frac{1}{2\sqrt{\cot(e^x)}} \cdot (-\csc^2(e^x)) \cdot \frac{d}{dx}e^x$$

ANSWER $\boxed{f'(x) = \dfrac{1}{2\sqrt{\cot(e^x)}} \cdot (-\csc^2(e^x)) \cdot e^x}$

(b) The derivative is obtained using the Quotient Rule and the fact that

$$\frac{d}{dx}\tan(x) = \sec^2(x) \quad \text{and} \quad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

We obtain

$$f'(t) = \frac{(\sqrt{t}+1) \cdot \frac{d}{dt}(t + \tan(t)) - (t + \tan(t)) \cdot \frac{d}{dt}(\sqrt{t}+1)}{(\sqrt{t}+1)^2}$$

ANSWER $\boxed{f'(t) = \dfrac{(\sqrt{t}+1) \cdot (1 + \sec^2(t)) - (t + \tan(t)) \cdot \left(\dfrac{1}{2\sqrt{t}} + 0\right)}{(\sqrt{t}+1)^2}}$

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Problem 4 Solution

4. Evaluate the limits

(a) $\displaystyle\lim_{x\to\infty} \frac{x^2 - x + 1}{\sqrt{x^4 + x}}$

(b) $\displaystyle\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$

Solution:

(a) We compute this limit by multiplying and dividing by $\frac{1}{x^2}$.

$$\lim_{x\to\infty} \frac{x^2 - x + 1}{\sqrt{x^4 + x}} = \lim_{x\to\infty} \frac{x^2 - x + 1}{\sqrt{x^4 + x}} \cdot \frac{1/x^2}{1/x^2}$$

$$\lim_{x\to\infty} \frac{x^2 - x + 1}{\sqrt{x^4 + x}} = \lim_{x\to\infty} \frac{1 - \frac{1}{x} + \frac{1}{x^2}}{\frac{1}{x^2}\sqrt{x^4 + 1}}$$

$$\lim_{x\to\infty} \frac{x^2 - x + 1}{\sqrt{x^4 + x}} = \lim_{x\to\infty} \frac{1 - \frac{1}{x} + \frac{1}{x^2}}{\sqrt{\frac{1}{x^4} \cdot (x^4 + 1)}}$$

$$\lim_{x\to\infty} \frac{x^2 - x + 1}{\sqrt{x^4 + x}} = \lim_{x\to\infty} \frac{1 - \frac{1}{x} + \frac{1}{x^2}}{\sqrt{1 + \frac{1}{x^4}}}$$

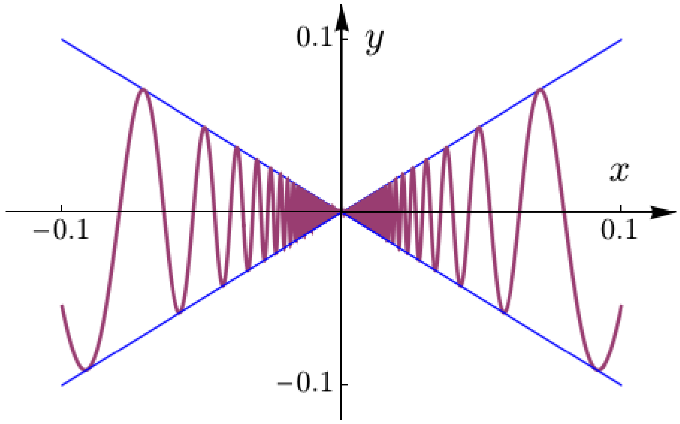
The value of the limit is obtained using the fact that

$$\lim_{x\to\infty} \frac{1}{x^n} = 0, \quad n > 0$$

We then obtain

Answer $$\boxed{\lim_{x\to\infty} \frac{x^2 - x + 1}{\sqrt{x^4 + x}} = \frac{1 - 0 + 0}{\sqrt{1 + 0}} = 1}$$

(b) First we identify the fact that the function $g(x) = \sin\left(\frac{1}{x}\right)$ fluctuates between -1 and 1 as $x \to 0$. Thus, the limit of this function does not exist as $x \to 0$. However, the function is bounded for all x while the function $f(x) = x$ tends to 0 as $x \to 0$. Therefore, the limit of the product $f(x)g(x) = x\sin\left(\frac{1}{x}\right)$ is 0.



To be more precise about the value of this limit, we use the Squeeze Theorem. To do this we begin by noting that

$$-|u| \leq u \leq |u|$$

for all u. By replacing u with the function $x \sin\left(\frac{1}{x}\right)$ we obtain

$$-\left|x \sin\left(\frac{1}{x}\right)\right| \leq x \sin\left(\frac{1}{x}\right) \leq \left|x \sin\left(\frac{1}{x}\right)\right|$$
$$-|x|\left|\sin\left(\frac{1}{x}\right)\right| \leq x \sin\left(\frac{1}{x}\right) \leq |x|\left|\sin\left(\frac{1}{x}\right)\right|$$

where we used the fact that $|ab| = |a||b|$. We now use the fact that

$$\left|\sin\left(\frac{1}{x}\right)\right| \leq 1$$

to obtain the inequality

$$-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$$

which is valid for all x. Furthermore, we know that

$$\lim_{x\to 0} (-|x|) = \lim_{x\to 0} |x| = 0$$

Thus, by the Squeeze Theorem we obtain

Answer $\boxed{\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0}$

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Problem 5 Solution

5. An object is thrown vertically upward. The position of the object after t seconds is given by the function $s(t) = -3t^2 + 2t + 1$ in the units of feet.

(a) Find the velocity and acceleration of the object at time t.

(b) What is the highest point the object will reach, and at what time?

(c) Calculate the point of time when the object hits the ground.

Solution:

(a) By definition, the velocity is $s'(t)$ and the acceleration is $s''(t)$. These derivatives are

$$\text{ANSWER} \quad \boxed{s'(t) = -6t + 2}$$

$$\text{ANSWER} \quad \boxed{s'(t) = -6}$$

(b) The object will reach its highest point when the velocity is zero. That is,

$$s'(t) = 0$$
$$-6t + 2 = 0$$
$$\text{ANSWER} \quad \boxed{t = \frac{1}{3}}$$

(c) The object will hit the ground when the position is zero. That is,

$$-3t^2 + 2t + 1 = 0$$
$$3t^2 - 2t - 1 = 0$$
$$(3t + 1)(t - 1) = 0$$
$$t = -\frac{1}{3}, \; t = 1$$

Since $t \geq 0$ we take the positive root. Therefore, the time when the object hits the ground is

$$\text{ANSWER} \quad \boxed{t = 1}$$