Math 180, Exam 1, Spring 2014
Problem 1 Solution

1. Find $\lim_{\theta \to 0} \dfrac{\tan\theta}{\theta}$.

Solution: The limit is indeterminate because, upon substituting $\theta = 0$, the function value tends toward $0/0$. This indeterminacy may be resolved by rewriting $\tan\theta$ as $\sin\theta/\cos\theta$ to yield

$$\begin{aligned}
\lim_{\theta \to 0} \frac{\tan\theta}{\theta} &= \lim_{\theta \to 0} \frac{\sin\theta/\cos\theta}{\theta} \\
\lim_{\theta \to 0} \frac{\tan\theta}{\theta} &= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{1}{\cos\theta} \\
\lim_{\theta \to 0} \frac{\tan\theta}{\theta} &= \left(\lim_{\theta \to 0} \frac{\sin\theta}{\theta}\right)\left(\lim_{\theta \to 0} \frac{1}{\cos\theta}\right) \\
\lim_{\theta \to 0} \frac{\tan\theta}{\theta} &= 1 \cdot \frac{1}{\cos 0} \\
\lim_{\theta \to 0} \frac{\tan\theta}{\theta} &= 1
\end{aligned}$$

If we let $x = 2\theta$, then the limit becomes

$$\lim_{\theta \to 0} \frac{\sin(2\theta)}{\theta} = 2 \cdot \lim_{x \to 0} \frac{\sin(x)}{x} = 2 \cdot 1 = 2$$

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Problem 2 Solution

2. Use the Squeeze Theorem to find $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x^3}\right)$.

Solution: Using the fact that $-1 \le \cos\theta \le 1$ for all θ we have the following inequalities:

$$
\begin{aligned}
-1 &\le \cos\theta \le 1 \\
-1 &\le \cos\left(\frac{1}{x^3}\right) \le 1 \\
-x^2 &\le x^2\cos\left(\frac{1}{x^3}\right) \le x^2
\end{aligned}
$$

for all $x \neq 0$. Since $\lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2 = 0$ we have, by the Squeeze Theorem,

$$\lim_{x\to 0} x^2 \cos\left(\frac{1}{x^3}\right) = 0$$

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Problem 3 Solution

3. Find all horizontal and vertical asymptotes of $f(x) = \dfrac{x^2+3x-4}{x^2-2x+1}$. Justify your answers using calculus.

Solution: The function may be rewritten as follows:

$$f(x) = \frac{x^2+3x-4}{x^2-2x+1} = \frac{(x-1)(x+4)}{(x-1)(x-1)}$$

- $x = 1$ is a **vertical asymptote** of f because

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{(x-1)(x+4)}{(x-1)(x-1)} \lim_{x\to 1^+} \frac{x+4}{x-1} = +\infty$$

- $y = 1$ is a **horizontal asymptote** of f because

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^2+3x-4}{x^2-2x+1} = \frac{1}{1} = 1$$

  where the ratio $1/1$ represents the ratio of the leading coefficients of the numerator and denominator, i.e. the coefficients of x.

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Problem 4 Solution

4. Differentiate $\dfrac{e^{-3x}}{5-x^2}$. Do not simplify your derivative.

Solution: Using the Quotient and Chain Rules, we have:

$$\frac{d}{dx}\left(\frac{e^{-3x}}{5-x^2}\right) = \frac{(5-x^2)\frac{d}{dx}e^{-3x} - e^{-3x}\frac{d}{dx}(5-x^2)}{(5-x^2)^2}$$

$$\frac{d}{dx}\left(\frac{e^{-3x}}{5-x^2}\right) = \frac{(5-x^2)\cdot(-3e^{-3x}) - e^{-3x}\cdot(-2x)}{(5-x^2)^2}$$

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Problem 5 Solution

5. Differentiate $e^{3t}\sqrt{\cot(2t)}$. Do not simplify your answer.

Solution: Using the Product and Chain Rules, we have:

$$\begin{aligned}
\frac{d}{dt}e^{3t}\sqrt{\cot(2t)} &= e^{3t}\frac{d}{dt}\sqrt{\cot(2t)} + \sqrt{\cot(2t)}\frac{d}{dt}e^{3t} \\
\frac{d}{dt}e^{3t}\sqrt{\cot(2t)} &= e^{3t}\cdot\frac{1}{2\sqrt{\cot(2t)}}\cdot\frac{d}{dt}\cot(2t) + \sqrt{\cot(2t)}\cdot(3e^{3t}) \\
\frac{d}{dt}e^{3t}\sqrt{\cot(2t)} &= e^{3t}\cdot\frac{1}{2\sqrt{\cot(2t)}}\cdot(-\csc^2(2t))\cdot\frac{d}{dt}(2t) + \sqrt{\cot(2t)}\cdot(3e^{3t}) \\
\frac{d}{dt}e^{3t}\sqrt{\cot(2t)} &= e^{3t}\cdot\frac{1}{2\sqrt{\cot(2t)}}\cdot(-\csc^2(2t))\cdot 2 + \sqrt{\cot(2t)}\cdot(3e^{3t})
\end{aligned}$$

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Problem 6 Solution

6. (a) Write the definition of the derivative of $f(x)$as the limit of a difference quotient.

(b) **Using the definition you wrote in part (a)**, find $f'(x)$ if $f(x) = \sqrt{x+1}$.

Solution: (a) The derivative of $f(x)$ is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(b) Using the above definition we have:

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
f'(x) &= \lim_{h \to 0} \frac{\sqrt{(x+h)+1} - \sqrt{x+1}}{h} \\
f'(x) &= \lim_{h \to 0} \frac{\sqrt{(x+h)+1} - \sqrt{x+1}}{h} \cdot \frac{\sqrt{(x+h)+1} + \sqrt{x+1}}{\sqrt{(x+h)+1} + \sqrt{x+1}} \\
f'(x) &= \lim_{h \to 0} \frac{((x+h)+1) - (x+1)}{h(\sqrt{(x+h)+1} + \sqrt{x+1})} \\
f'(x) &= \lim_{h \to 0} \frac{h}{h(\sqrt{(x+h)+1} + \sqrt{x+1})} \\
f'(x) &= \lim_{h \to 0} \frac{1}{\sqrt{(x+h)+1} + \sqrt{x+1}} \\
f'(x) &= \frac{1}{\sqrt{(x+0)+1} + \sqrt{x+1}} \\
f'(x) &= \frac{1}{2\sqrt{x+1}}
\end{aligned}$$

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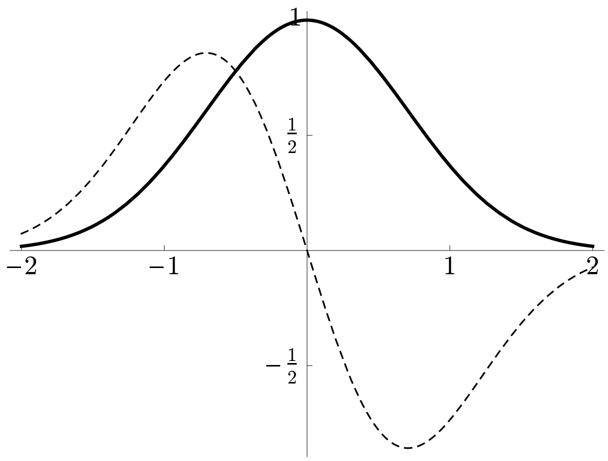
Problem 7 Solution

7. The following picture shows two functions, y_1 (solid curve) and y_2 (dashed curve). One of the functions is the derivative of the other. Determine which function is the derivative of the other and give three examples/reasons why your choice is correct.



Solution: $y_2 = y_1'$

(1) $y_2(0) = y_1'(0) = 0$

(2) $y_2 > 0$ and $y_1' > 0$ for $x < 0$

(3) $y_2 < 0$ and $y_1' < 0$ for $x > 0$

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Problem 8 Solution

8. On the axes provided, sketch a possible graph for f with the following information.

(1) $f(0) = -1$

(2) $\lim_{x \to 2^-} f(x) = -\infty$

(3) $\lim_{x \to 2^+} f(x) = +\infty$

(4) $\lim_{x \to -\infty} f(x) = 0$

(5) $\lim_{x \to +\infty} f(x) = 2$