Math 180, Exam 2, Fall 2007
Problem 1 Solution

1. Differentiate the following functions:

   (a) $x^2 \ln x$

   (b) $\sin(a + bx)$

   (c) $\arctan(3x)$

Solution:

(a) Use the Product Rule.

$$\begin{aligned}(x^2 \ln x)' &= (x^2)(\ln x)' + (x^2)'(\ln x)\\ &= (x^2)\left(\frac{1}{x}\right) + (2x)(\ln x)\\ &= \boxed{x + 2x \ln x}\end{aligned}$$

(b) Use the Chain Rule.

$$\begin{aligned}[\sin(a+bx)]' &= \cos(a+bx) \cdot (a+bx)'\\ &= \boxed{b\cos(a+bx)}\end{aligned}$$

(c) Use the Chain Rule.

$$\begin{aligned}[\arctan(3x)]' &= \frac{1}{1+(3x)^2} \cdot (3x)'\\ &= \boxed{\frac{3}{1+9x^2}}\end{aligned}$$

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Problem 2 Solution

2. Differentiate the following functions:

(a) e^{2-x^2}

(b) $x\cos(x)$

(c) $\arcsin(x/2)$

Solution:

(a) Use the Chain Rule.

$$\begin{aligned}\left(e^{2-x^2}\right)' &= e^{2-x^2} \cdot (2-x^2)' \\ &= \boxed{-2xe^{2-x^2}}\end{aligned}$$

(b) Use the Product Rule.

$$\begin{aligned}[x\cos(x)]' &= (x)(\cos x)' + (x)'(\cos x) \\ &= \boxed{-x\sin x + \cos x}\end{aligned}$$

(c) Use the Chain Rule.

$$\begin{aligned}[\arcsin(x/2)]' &= \frac{1}{\sqrt{1-(x/2)^2}} \cdot (x/2)' \\ &= \frac{1}{\sqrt{1-x^2/4}} \cdot \frac{1}{2} \\ &= \boxed{\frac{1}{\sqrt{4-x^2}}}\end{aligned}$$

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Problem 3 Solution

3. Let $y = f(x)$ be the function defined implicitly by $y^3 - y + x = 0$ and $f(6) = -2$.

(a) Find $\dfrac{dy}{dx}$ at the point $(6, -2)$.

(b) Find the equation of the tangent line at $(6, -2)$.

Solution:

(a) To find $\dfrac{dy}{dx}$, we use implicit differentiation.

$$\begin{aligned} y^3 - y + x &= 0 \\ \frac{d}{dx}y^3 - \frac{d}{dx}y + \frac{d}{dx}x &= \frac{d}{dx}0 \\ 3y^2\frac{dy}{dx} - \frac{dy}{dx} + 1 &= 0 \\ 3y^2\frac{dy}{dx} - \frac{dy}{dx} &= -1 \\ \frac{dy}{dx}(3y^2 - 1) &= -1 \\ \frac{dy}{dx} &= \frac{-1}{3y^2 - 1} \end{aligned}$$

At the point $(6, -2)$, the value of $\dfrac{dy}{dx}$ is:

$$\left.\frac{dy}{dx}\right|_{(6,-2)} = \frac{-1}{3(-2)^2 - 1} = \boxed{-\frac{1}{11}}$$

(b) The value of $\dfrac{dy}{dx}$ from part (a) is the slope of the tangent line at the point $(6, -2)$. An equation for the tangent line is then:

$$\boxed{y + 2 = -\frac{1}{11}(x - 6)}$$

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Problem 4 Solution

4. Use the information in the table about f and g to find:

(a) $h'(0)$, where $h(x) = f(g(x))$

(b) $k'(2)$, where $k(x) = f(x)g(x)$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	1	-1	2	5
1	-1	2	4	3
2	7	3	1	4

Solution:

(a) Use the Chain Rule.

$$h'(x) = [f(g(x))]' = f'(g(x))g'(x)$$

Now plug in $x = 0$ and use the table.

$$\begin{aligned} h'(0) &= f'(g(0))g'(0) \\ &= f'(2) \cdot 5 \\ &= 3 \cdot 5 \\ &= \boxed{15} \end{aligned}$$

(b) Use the Product Rule.

$$k'(x) = [f(x)g(x)]' = f(x)g'(x) + f'(x)g(x)$$

Now plug in $x = 2$ and use the table.

$$\begin{aligned} k'(2) &= f(2)g'(2) + f'(2)g(2) \\ &= (7)(4) + (3)(1) \\ &= \boxed{31} \end{aligned}$$

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Problem 5 Solution

5. Find the critical points of the function $f(x) = x^3 + 3x^2 - 9x - 11$ and find the global minimum of $f(x)$ on the interval $-4 \leq x \leq 3$.

Solution: The critical points are the values of x for which either $f'(x) = 0$ or $f'(x)$ does not exist. Since $f(x)$ is a polynomial, $f'(x)$ exists for all $x \in \mathbb{R}$. Therefore, the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(x^3 + 3x^2 - 9x - 11\right)' &= 0 \\ 3x^2 + 6x - 9 &= 0 \\ 3(x^2 + 2x - 3) &= 0 \\ 3(x+3)(x-1) &= 0 \\ x = -3,\ x &= 1 \end{aligned}$$

The critical points are $\boxed{x = -3, 1}$.

The global minimum of $f(x)$ will occur at a critical point in the interval $[-4, 3]$ or at one of the endpoints. The critical points $x = -3$ and $x = 1$ both lie in $[-4, 3]$. Therefore, we check the value of $f(x)$ at $x = -4, -3, 1,$ and 3.

$$\begin{aligned} f(-4) &= (-4)^3 + 3(-4)^2 - 9(-4) - 11 = 9 \\ f(-3) &= (-3)^3 + 3(-2)^2 - 9(-3) - 11 = 16 \\ f(1) &= 1^3 + 3(1)^2 - 9(1) - 11 = -16 \\ f(3) &= 3^3 + 3(3)^2 - 9(3) - 11 = 16 \end{aligned}$$

The global minimum of $f(x)$ on $[-4, 3]$ is $\boxed{-16}$ because it is the smallest of the above values of f.

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Problem 6 Solution

6. Find the x- and y-coordinates of all local maxima, local minima, and inflection points of $f(x) = x^3 - 3x + 2$.

Solution: The critical points of $f(x)$ are the values of x for which either $f'(x)$ does not exist or $f'(x) = 0$. Since $f(x)$ is a polynomial, $f'(x)$ exists for all $x \in \mathbb{R}$. Therefore, the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(x^3 - 3x + 2\right)' &= 0 \\ 3x^2 - 3 &= 0 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

We will use the Second Derivative Test to classify the critical points $\boxed{x = \pm 1}$. The second derivative is $f''(x) = 6x$. The values of $f''(x)$ at the critical points are:

$$\begin{aligned} f''(-1) &= 6(-1) = -6 \\ f''(1) &= 6(1) = 6 \end{aligned}$$

Since $f''(-1) < 0$ the Second Derivative Test implies that $\boxed{f(-1) = 4}$ is a local maximum and since $f''(1) > 0$ the Second Derivative Test implies that $\boxed{f(1) = 0}$ is a local minimum.

Possible inflection points are the values of x for which $f''(x) = 0$. Since $f''(x) = 6x$, the only possible inflection point is $x = 0$. In order for $x = 0$ to be an inflection point, there must be a sign change in $f''(x)$ at $x = 0$. We have already shown that $f''(-1) < 0$ and $f''(1) > 0$ which implies a sign change at $x = 0$. Therefore, $x = 0$ is an inflection point. The corresponding y-coordinate is $f(0) = 2$.

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Problem 7 Solution

7. Find $\lim\limits_{x \to 0} \dfrac{1 - e^x}{x - x^2}$.

Solution: Upon substituting $x = 0$ into the function $\frac{1-e^x}{x-x^2}$ we find that

$$\frac{1 - e^0}{0 - 0^2} = \frac{0}{0}$$

which is indeterminate. We resolve the indeterminacy using L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x \to 0} \frac{1 - e^x}{x - x^2} &\overset{\text{L'H}}{=} \lim_{x \to 0} \frac{(1 - e^x)'}{(x - x^2)'} \\
&= \lim_{x \to 0} \frac{-e^x}{1 - 2x} \\
&= \frac{-e^0}{1 - 2(0)} \\
&= \boxed{-1}
\end{aligned}$$

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Problem 8 Solution

8. You wish to enclose a 400 square-foot rectangular garden with shrubs costing \$40 per foot on the three sides and a wall costing \$20 per foot on the fourth side. Find the dimensions that minimize the total cost.

Solution: We begin by letting x be the length of the wall and the length of the side with shrubs opposite the wall and letting y be the lengths of the remaining two sides with shrubs. The function we seek to minimize is the cost:

$$\textbf{Function}: \quad \text{Cost} = \$20x + \$40x + \$40y + \$40y \tag{1}$$

where $\$20x$ is the cost of making the wall and the remaining terms are the costs of the shrubs. The constraint in this problem is that the area of the garden is 400 square feet.

$$\textbf{Constraint}: \quad xy = 400 \tag{2}$$

Solving the constraint equation (2) for y we get:

$$y = \frac{400}{x} \tag{3}$$

Plugging this into the function (1) and simplifying we get:

$$\begin{aligned} \text{Cost} &= \$20x + \$40x + \$40\left(\frac{400}{x}\right) + \$40\left(\frac{400}{x}\right) \\ f(x) &= 60x + \frac{32000}{x} \end{aligned}$$

We want to find the absolute minimum of $f(x)$ on the **interval** $(0, \infty)$. We choose this interval because x must be nonnegative (x represents a length) and non-zero (if x were 0, then the area would be 0 but it must be 400).

The absolute minimum of $f(x)$ will occur either at a critical point of $f(x)$ in $(0, \infty)$ or it will not exist because the interval is open. The critical points of $f(x)$ are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(60x + \frac{32000}{x}\right)' &= 0 \\ 60 - \frac{32000}{x^2} &= 0 \\ 60x^2 - 32000 &= 0 \\ x^2 &= \frac{32000}{60} \\ x &= \pm\frac{40}{\sqrt{3}} \end{aligned}$$

However, since $x = -\frac{40}{\sqrt{3}}$ is outside $(0, \infty)$, the only critical point is $x = \frac{40}{\sqrt{3}}$. Plugging this into $f(x)$ we get:

$$f\left(\frac{40}{\sqrt{3}}\right) = 60\left(\frac{40}{\sqrt{3}}\right) + \frac{32000}{\frac{40}{\sqrt{3}}} = 1600\sqrt{3}$$

Taking the limits of $f(x)$ as x approaches the endpoints we get:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(60x + \frac{32000}{x}\right) = 0 + \infty = \infty$$
$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty}\left(60x + \frac{32000}{x}\right) = \infty + 0 = \infty$$

both of which are larger than $1600\sqrt{3}$. We conclude that the cost is an absolute minimum at $\boxed{x = \frac{40}{\sqrt{3}}}$ and that the resulting cost is \$$1600\sqrt{3}$. The last step is to find the corresponding value for y by plugging $x = \frac{40}{\sqrt{3}}$ into equation (3).

$$y = \frac{400}{x}$$
$$y = \frac{400}{40/\sqrt{3}}$$
$$\boxed{y = 10\sqrt{3}}$$

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Problem 9 Solution

9. Find the largest interval on which $f(x) = (x^2 + 1)e^{-x}$ is concave down.

Solution: We begin by computing $f'(x)$ using the Product and Chain Rules.

$$\begin{aligned}
f'(x) &= \left[(x^2 + 1)e^{-x}\right]' \\
f'(x) &= (x^2 + 1)(e^{-x})' + (e^{-x})(x^2 + 1)' \\
f'(x) &= -(x^2 + 1)e^{-x} + 2xe^{-x} \\
f'(x) &= (-x^2 + 2x - 1)e^{-x}
\end{aligned}$$

We now compute $f''(x)$ using the Product and Chain Rules.

$$\begin{aligned}
f''(x) &= \left[(-x^2 + 2x - 1)e^{-x}\right]' \\
f''(x) &= (-x^2 + 2x - 1)(e^{-x})' + (e^{-x})(-x^2 + 2x - 1)' \\
f''(x) &= -(-x^2 + 2x - 1)e^{-x} + (-2x + 2)e^{-x} \\
f''(x) &= e^{-x}(x^2 - 4x + 3)
\end{aligned}$$

To determine where $f(x)$ is concave down we find all solutions to $f''(x) = 0$.

$$\begin{aligned}
f''(x) &= 0 \\
e^{-x}(x^2 - 4x + 3) &= 0 \\
x^2 - 4x + 3 &= 0 \\
(x - 1)(x - 3) &= 0 \\
x &= 1, \ x = 3
\end{aligned}$$

The domain of $f(x)$ is $(-\infty, \infty)$. We now split the domain into the three intervals $(-\infty, 1)$, $(1, 3)$, and $(3, \infty)$. We then evaluate $f''(x)$ at a test point in each interval.

Interval	**Test Point**, c	$f''(c)$	**Sign of** $f''(c)$
$(-\infty, 1)$	0	$f''(0) = 3$	$+$
$(1, 3)$	2	$f''(2) = -e^{-2}$	$-$
$(3, \infty)$	4	$f''(4) = 3e^{-4}$	$+$

Using the table we conclude that $f(x)$ is concave down on $\boxed{(1, 3)}$ because $f''(x) < 0$ for all $x \in (1, 3)$.

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Problem 10 Solution

10. The graph below is of the *derivative* $f'(x)$ on the interval $(-0.5, 5.5)$. Determine the intervals on which the original function f is:

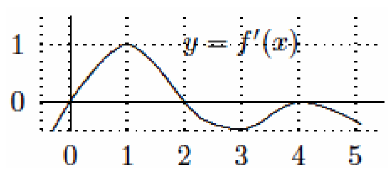
(a) increasing,

(b) decreasing,

(c) concave up,

(d) concave down.

(e) Give one value of x at which f has a local maximum.



Solution:

(a) $f(x)$ is increasing on (a, b) when $f'(x) > 0$ for all $x \in (a, b)$. This occurs on $(0, 2)$ because the graph is above the x-axis for these values of x.

(b) $f(x)$ is decreasing on (a, b) when $f'(x) < 0$ for all $x \in (a, b)$. This occurs on $(-0.5, 0) \cup (2, 4) \cup (4, 5.5)$ because the graph is below the x-axis for these values of x.

(c) $f(x)$ is concave up on (a, b) when $f'(x)$ is increasing on (a, b). This occurs on $(-0.5, 1) \cup (3, 4)$ because the graph is rising for these values of x.

(d) $f(x)$ is concave down on (a, b) when $f'(x)$ is decreasing on (a, b). This occurs on $(1, 3) \cup (4, 5.5)$ because the graph is falling for these values of x.

(e) f has a local maximum at $x = c$ when $f'(c) = 0$ and the sign of f' changes from positive to negative at $x = c$. This occurs at $x = 2$.