Math 180, Exam 2, Fall 2008
Problem 1 Solution

1. Find the derivatives of the following functions, do simplify.

(a) $\ln(x^2 + x + 1)$, (b) $\cos(\sqrt{x})$, (c) $\arctan(x)$

Solution:

(a) Use the Chain Rule.

$$\begin{aligned}[\ln(x^2 + x + 1)]' &= \frac{1}{x^2 + x + 1} \cdot (x^2 + x + 1)' \\ &= \boxed{\frac{1}{x^2 + x + 1} \cdot (2x + 1)}\end{aligned}$$

(b) Use the Chain Rule.

$$\begin{aligned}\left[\cos\left(\sqrt{x}\right)\right]' &= -\sin\left(\sqrt{x}\right) \cdot \left(\sqrt{x}\right)' \\ &= \boxed{-\sin\left(\sqrt{x}\right) \cdot \frac{1}{2\sqrt{x}}}\end{aligned}$$

(c) This is a basic derivative that you should know.

$$[\arctan(x)]' = \boxed{\frac{1}{1 + x^2}}$$

Math 180, Exam 2, Fall 2008
Problem 2 Solution

2. Find the derivatives $f'(x)$ and $f''(x)$ for the function $f(x) = e^{-x}\sin(x)$.

Solution: The first derivative $f'(x)$ is found using the Product and Chain Rules.

$$\begin{aligned} f'(x) &= \left[e^{-x}\sin(x)\right]' \\ &= e^{-x}(\sin(x))' + \sin(x)(e^{-x})' \\ &= e^{-x}\cos(x) + \sin(x)(-e^{-x}) \\ &= \boxed{e^{-x}(\cos(x) - \sin(x))} \end{aligned}$$

The second derivative $f''(x)$ is found using the Product and Chain Rules.

$$\begin{aligned} f''(x) &= [f'(x)]' \\ &= [e^{-x}(\cos(x) - \sin(x))]' \\ &= e^{-x}(\cos(x) - \sin(x))' + (\cos(x) - \sin(x))(e^{-x})' \\ &= e^{-x}(-\sin(x) - \cos(x)) + (\cos(x) - \sin(x))(-e^{-x}) \\ &= \boxed{-2e^{-x}\cos(x)} \end{aligned}$$

Math 180, Exam 2, Fall 2008
Problem 3 Solution

3. Use implicit differentiation to find the slope of the line tangent to the curve

$$xy^2 + 2x^2 - y = 0$$

at the point $(-1, 1)$.

Solution: Using implicit differentiation we get:

$$\begin{aligned}
xy^2 + 2x^2 - y &= 0 \\
(xy^2)' + (2x^2)' - (y)' &= (0)' \\
\big[(x)(y^2)' + (y^2)(x)'\big] + 4x - y' &= 0 \\
\big[(x)(2yy') + (y^2)(1)\big] + 4x - y' &= 0 \\
2xyy' + y^2 + 4x - y' &= 0 \\
2xyy' - y' &= -y^2 - 4x \\
y'(2xy - 1) &= -y^2 - 4x \\
y' &= \frac{-y^2 - 4x}{2xy - 1}
\end{aligned}$$

At the point $(-1, 1)$, the value of y' is:

$$y'(-1, 1) = \frac{-(1)^2 - 4(-1)}{2(-1)(1) - 1} = \boxed{-1}$$

Math 180, Exam 2, Fall 2008
Problem 4 Solution

4. Let $f(x) = x^4 - 6x^2 + 2$.

(a) Find the critical points and the inflection points of f.

(b) On what interval is f concave down?

(c) Find the minimum value of f.

Solution:

(a) The critical points of $f(x)$ are the values of x for which either $f'(x)$ does not exist or $f'(x) = 0$. Since $f(x)$ is a polynomial, $f'(x)$ exists for all $x \in \mathbb{R}$ so the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(x^4 - 6x^2 + 2\right)' &= 0 \\ 4x^3 - 12x &= 0 \\ 4x(x^2 - 3) &= 0 \\ x = 0,\ x &= \pm\sqrt{3} \end{aligned}$$

Thus, $\boxed{x = 0}$ and $\boxed{x = \pm\sqrt{3}}$ are the critical points of f.

The inflection points of $f(x)$ are the values of x where a sign change in $f''(x)$ occurs. To determine these points, we start by finding the solutions to $f''(x) = 0$.

$$\begin{aligned} f''(x) &= 0 \\ \left(4x^3 - 12x\right)' &= 0 \\ 12x^2 - 12 &= 0 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

We now split the domain $(-\infty, \infty)$ into the three intervals $(-\infty, -1)$,$(-1, 1)$, and $(1, \infty)$. We then evaluate $f''(x)$ at a test point in each interval.

Interval	**Test Point**, c	$f'(c)$	**Sign of** $f'(c)$
$(-\infty, -1)$	-2	$f''(-2) = 36$	$+$
$(-1, 1)$	0	$f''(0) = -12$	$-$
$(1, \infty)$	2	$f''(2) = 36$	$+$

Since there are sign changes in $f''(x)$ at both $x = \pm 1$, the points $\boxed{x = \pm 1}$ are inflection points.

(b) From the table above, we conclude that f is concave down on $\boxed{(-1, 1)}$ because $f''(x) < 0$ for all $x \in (-1, 1)$.

(c) The domain of $f(x)$ is $(-\infty, \infty)$. As $x \to \pm\infty$, $f(x) \to \infty$. Therefore, the absolute minimum of $f(x)$ will occur at a critical point. Evaluating $f(x)$ at $x = 0, \pm\sqrt{3}$ we get:

$$\begin{aligned} f(0) &= 2 \\ f\left(\sqrt{3}\right) &= -7 \\ f\left(-\sqrt{3}\right) &= -7 \end{aligned}$$

Thus, the absolute minimum value of $f(x)$ is $\boxed{-7}$.

Math 180, Exam 2, Fall 2008
Problem 5 Solution

5. Find the limit: $\lim_{x\to 1} \frac{\ln(x)}{x^3 - 1}$.

Solution: Upon substituting $x = 1$ into the function $\frac{\ln(x)}{x^3-1}$ we find that

$$\frac{\ln(1)}{1^3 - 1} = \frac{0}{0}$$

which is indeterminate. We resolve the indeterminacy using L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to 1} \frac{\ln(x)}{x^3 - 1} &\overset{\text{L'H}}{=} \lim_{x\to 1} \frac{(\ln(x))'}{(x^3 - 1)'} \\
&= \lim_{x\to 1} \frac{\frac{1}{x}}{3x^2} \\
&= \lim_{x\to 1} \frac{1}{3x^3} \\
&= \frac{1}{3(1)^3} \\
&= \boxed{\frac{1}{3}}
\end{aligned}$$

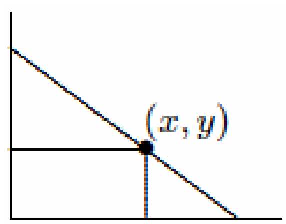
Math 180, Exam 2, Fall 2008
Problem 6 Solution

6. A family of rectangles in the xy-plane has one side on the x-axis, the lower left corner at the origin $(0,0)$, and the upper right corner at a point (x,y) on the straight line

$$3x + 4y = 5.$$

(a) Find the area of such a rectangle as a function of x alone.

(b) Find the dimensions, x and y, of the particular rectangle with the largest area.



Solution:

(a) The dimensions of the rectangle are x and y. Therefore, the area of the rectangle has the equation:

$$\text{Area} = xy \tag{1}$$

We are asked to write the area as a function of x alone. Therefore, we must find an equation that relates x to y so that we can eliminate y from the area equation. This equation is

$$3x + 4y = 5 \tag{2}$$

because (x,y) must lie on this line. Solving equation (2) for y we get:

$$y = \frac{5}{4} - \frac{3}{4}x \tag{3}$$

Plugging this into the area equation we get:

$$\text{Area} = x\left(\frac{5}{4} - \frac{3}{4}x\right)$$

$$\boxed{f(x) = \frac{5}{4}x - \frac{3}{4}x^2}$$

(b) We seek the value of x that maximizes $f(x)$. The interval in the problem is $[0, \frac{5}{3}]$ because the upper corner of the rectangle must lie in the first quadrant.

The absolute maximum of $f(x)$ will occur either at a critical point of $f(x)$ in $[0, \frac{5}{3}]$ or at one of the endpoints. The critical points of $f(x)$ are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(\frac{5}{4}x - \frac{3}{4}x^2\right)' &= 0 \\ \frac{5}{4} - \frac{3}{2}x &= 0 \\ 5 - 6x &= 0 \\ \boxed{x = \frac{5}{6}} \end{aligned}$$

Plugging this into $f(x)$ we get:

$$f\left(\frac{5}{6}\right) = \frac{5}{4}\left(\frac{5}{6}\right) - \frac{3}{4}\left(\frac{5}{6}\right)^2 = \frac{25}{48}$$

Evaluating $f(x)$ at the endpoints $x = 0$ and $x = \frac{5}{3}$ we get:

$$\begin{aligned} f(0) &= \frac{5}{4}(0) - \frac{3}{4}(0)^2 = 0 \\ f\left(\frac{5}{3}\right) &= \frac{5}{4}\left(\frac{5}{3}\right) - \frac{3}{4}\left(\frac{5}{3}\right)^2 = 0 \end{aligned}$$

both of which are smaller than $\frac{25}{48}$. We conclude that the area is an absolute maximum at $x = \frac{5}{6}$ and that the resulting area is $\frac{25}{48}$. The last step is to find the corresponding value for y by plugging $x = 3$ into equation (3).

$$y = \frac{5}{4} - \frac{3}{4}\left(\frac{5}{6}\right) = \boxed{\frac{5}{8}}$$