Math 180, Exam 2, Fall 2009
Problem 1 Solution

1. Differentiate with respect to x. Do not simplify your answers.

(a) $\dfrac{\sin(2x)}{\cos(3x)}$, (b) $\sqrt{x^2 - 7x + 1}$, (c) $\arctan(3x^3)$

Solution:

(a) Use the Quotient and Chain Rules.

$$\left[\frac{\sin(2x)}{\cos(3x)}\right]' = \frac{\cos(3x)[\sin(2x)]' - \sin(2x)[\cos(3x)]'}{[\cos(3x)]^2}$$
$$= \frac{\cos(3x)\cos(2x)\cdot(2x)' - \sin(2x)[-\sin(3x)]\cdot(3x)'}{[\cos(3x)]^2}$$
$$= \boxed{\frac{\cos(3x)\cos(2x)\cdot 2 + \sin(2x)\sin(3x)\cdot 3}{\cos^2(3x)}}$$

(b) Use the Chain Rule.

$$\left(\sqrt{x^2 - 7x + 1}\right)' = \frac{1}{2}\left(x^2 - 7x + 1\right)^{-1/2}\cdot(x^2 - 7x + 1)'$$
$$= \boxed{\frac{1}{2}\left(x^2 - 7x + 1\right)^{-1/2}\cdot(2x - 7)}$$

(c) Use the Chain Rule.

$$\left[\arctan(3x^3)\right]' = \frac{1}{1 + (3x^3)^2}\cdot(3x^3)'$$
$$= \boxed{\frac{1}{1 + (3x^3)^2}\cdot 9x^2}$$

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Problem 2 Solution

2. The table below gives values for f and g and their derivatives:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	3	-2	8	4
1	-1	2	5	-3
2	5	3	1	5

(a) Find $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right)$ at $x = 1$.

(b) Find $\dfrac{d}{dx}f(g(x))$ at $x = 2$.

(c) Find $\dfrac{d}{dx}\ln(3f(x))$ at $x = 0$.

Solution:

(a) Use the Product Rule.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

At $x = 1$ we have:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)\bigg|_{x=1} &= \frac{g(1)f'(1) - f(1)g'(1)}{g(1)^2} \\
&= \frac{(5)(2) - (-1)(-3)}{5^2} \\
&= \boxed{\frac{7}{25}}
\end{aligned}$$

(b) Use the Chain Rule.

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

At $x = 2$ we have:

$$\begin{aligned}
\frac{d}{dx}f(g(x))\bigg|_{x=2} &= f'(g(2))g'(2) \\
&= f'(1)g'(2) \\
&= (2)(5) \\
&= \boxed{10}
\end{aligned}$$

(c) Use the Chain Rule.

$$\begin{aligned}\frac{d}{dx}\ln(3f(x)) &= \frac{1}{3f(x)}\cdot(3f(x))' \\ &= \frac{1}{3f(x)}\cdot 3f'(x) \\ &= \frac{f'(x)}{f(x)}\end{aligned}$$

At $x = 0$ we have:

$$\begin{aligned}\frac{d}{dx}\ln(3f(x))\bigg|_{x=0} &= \frac{f'(0)}{f(0)} \\ &= \boxed{\frac{-2}{3}}\end{aligned}$$

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Problem 3 Solution

3. Suppose x and y are related by the equation $xy^3 + \tan(y) + x^3 = 27$.

(a) Find $\dfrac{dy}{dx}$ in terms of x and y.

(b) Let f be a function where $y = f(x)$ satisfies this equation and where $f(3) = 0$. Use the linearization of f to approximate $f(3.1)$.

Solution:

(a) We find $\dfrac{dy}{dx}$ using implicit differentiation.

$$\begin{aligned}
xy^3 + \tan(y) + x^3 &= 27 \\
\frac{d}{dx}xy^3 + \frac{d}{dx}\tan(y) + \frac{d}{dx}x^3 &= \frac{d}{dx}27 \\
x\frac{d}{dx}y^3 + y^3\frac{d}{dx}x + \sec^2(y)\frac{dy}{dx} + 3x^2 &= 0 \\
x\left(3y^2\frac{dy}{dx}\right) + y^3(1) + \sec^2(y)\frac{dy}{dx} + 3x^2 &= 0 \\
3xy^2\frac{dy}{dx} + \sec^2(y)\frac{dy}{dx} &= -y^3 - 3x^2 \\
\frac{dy}{dx}(3xy^2 + \sec^2(y)) &= -y^3 - 3x^2 \\
\boxed{\frac{dy}{dx} = \frac{-y^3 - 3x^2}{3xy^2 + \sec^2(y)}}
\end{aligned}$$

(b) The linearization of $y = f(x)$ at $x = 3$ is:

$$L(x) = f(3) + f'(3)(x - 3)$$

where $f(3) = 0$ and

$$\begin{aligned}
f'(3) &= \left.\frac{dy}{dx}\right|_{(3,0)} \\
&= \frac{-0^2 - 3(3)^2}{3(3)(0)^2 + \sec^2 0} \\
&= -27
\end{aligned}$$

Therefore, the linearization is $L(x) = 0 - 27(x - 3) = -27(x - 3)$. The approximate value of $f(3.1)$ is $L(3.1)$:

$$L(3.1) = -27(3.1 - 3) = \boxed{-2.7}$$

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Problem 4 Solution

4. Suppose that a function $f(x)$ is defined and is decreasing and concave down for all x. Also $f(3) = 5$ and $f'(3) = -2$.

(a) Using the given properties of f, find an integer n with $|f(2) - n| < 1$.

(b) If $f(r) = 0$, find an integer k with $|r - k| < 2$.

Solution:

(a) An equation for the line tangent to $y = f(x)$ at $x = 3$ is:

$$y - 5 = -2(x - 3)$$
$$y = -2x + 11$$

When $x = 2$, we have $y = -2(2) + 11 = 7$. Thus, $(2, 7)$ is a point on the tangent line.

Knowing that f is decreasing and concave down for all x we know that the tangent line sits above the graph of $y = f(x)$ for all $x \neq 3$. Therefore, $f(2) < 7$. Furthermore, since f is decreasing we know that $f(2) > f(3) = 5$. We then conclude that $f(2)$ satisfies the inequality:

$$5 < f(2) < 7$$
$$-1 < f(2) - 6 < 1$$
$$|f(2) - 6| < 1$$

Therefore, we choose $\boxed{n = 6}$ to satisfy the inequality $|f(2) - n| < 1$.

(b) The tangent line intersects the x-axis at $x = \frac{11}{2} = 5.5$. Using the fact that the tangent line sits above the graph of $y = f(x)$ for $x \neq 3$, we know that:

$$-2r + 11 > f(r)$$
$$-2r + 11 > 0$$
$$2r < 11$$
$$r < \frac{11}{2} = 5.5$$

Since we're given that $f(3) = 5$ and that f is decreasing for all x we know that $r > 3$. Therefore, r must satisfy the inequality $3 < r < 5.5$. To find an integer k that satisfies $|r - k| < 2$, we manipulate the inequality $3 < r < 5.5$ as follows:

$$3 < r < 5.5$$
$$2 < 3 < r < 5.5 < 6$$
$$2 < r < 6$$
$$-2 < r - 4 < 2$$
$$|r - 4| < 2$$

Therefore, we choose $\boxed{k = 4}$.

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Problem 5 Solution

5. Suppose b is a positive real number, and consider the function

$$f(x) = 3e^{-x^2/b}$$

(a) Find the x-coordinates of the inflection points of $f(x)$.

(b) Is the graph of $f(x)$ concave up or concave down for x near 0?

Solution:

(a) We begin by computing $f'(x)$ using the Chain Rule.

$$\begin{aligned}
f'(x) &= \left(3e^{-x^2/b}\right)' \\
f'(x) &= 3e^{-x^2/b} \cdot \left(-x^2/b\right)' \\
f'(x) &= 3e^{-x^2/b} \cdot \left(-\frac{2}{b}x\right) \\
f'(x) &= -\frac{6}{b}xe^{-x^2/b}
\end{aligned}$$

We now compute $f''(x)$ using the Product and Chain Rules.

$$\begin{aligned}
f''(x) &= \left(-\frac{6}{b}xe^{-x^2/b}\right)' \\
f''(x) &= \left(-\frac{6}{b}x\right)\left(e^{-x^2/b}\right)' + \left(e^{-x^2/b}\right)\left(-\frac{6}{b}x\right)' \\
f''(x) &= \left(-\frac{6}{b}x\right)\left(-\frac{2}{b}xe^{-x^2/b}\right) + \left(e^{-x^2/b}\right)\left(-\frac{6}{b}\right) \\
f''(x) &= \frac{6}{b}e^{-x^2/b}\left(\frac{2}{b}x^2 - 1\right)
\end{aligned}$$

The inflection points of $f(x)$ are the points where $f''(x)$ changes sign. For the given function $f(x)$, the critical points will occur when $f''(x) = 0$. The solutions to this equation are:

$$\begin{aligned}
f''(x) &= 0 \\
\frac{6}{b}e^{-x^2/b}\left(\frac{2}{b}x^2 - 1\right) &= 0 \\
\frac{2}{b}x^2 - 1 &= 0 \\
x^2 &= \frac{b}{2} \\
\boxed{x = \pm\sqrt{\frac{b}{2}}}
\end{aligned}$$

The domain of $f(x)$ is $(-\infty, \infty)$. We now split the domain into the three intervals $(-\infty, -\sqrt{\frac{b}{2}})$, $(-\sqrt{\frac{b}{2}}, \sqrt{\frac{b}{2}})$, and $(\sqrt{\frac{b}{2}}, \infty)$. We then evaluate $f''(x)$ at a test point in each interval.

Interval	**Test Point**, c	$f''(c)$	**Sign of** $f''(c)$
$(-\infty, -\sqrt{\frac{b}{2}})$	$-\sqrt{b}$	$f''(-\sqrt{b}) = \frac{6}{b}e^{-1}$	$+$
$(-\sqrt{\frac{b}{2}}, \sqrt{\frac{b}{2}})$	0	$f''(0) = -\frac{6}{b}$	$-$
$(\sqrt{\frac{b}{2}}, \infty)$	$\sqrt{b}$	$f''(\sqrt{b}) = \frac{6}{b}e^{-1}$	$+$

Since $f''(x)$ changes sign at $x = \pm\sqrt{\frac{b}{2}}$, the points $x = \pm\sqrt{\frac{b}{2}}$ are inflection points.

(b) Using the table we conclude that $f(x)$ is **concave down** on $(-\sqrt{\frac{b}{2}}, \sqrt{\frac{b}{2}})$ because $f''(x) < 0$ for all $x \in (-\sqrt{\frac{b}{2}}, \sqrt{\frac{b}{2}})$.

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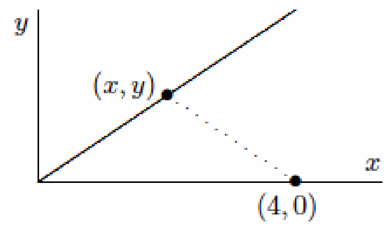
Problem 6 Solution

6. Find the point (x, y) on the line $y = \frac{3}{4}x$ closest to the point $(4, 0)$.



Solution: The function we seek to minimize is the distance between (x, y) and $(4, 0)$.

Function : $\quad \text{Distance} = \sqrt{(x-4)^2 + (y-0)^2} \qquad (1)$

The constraint in this problem is that the point (x, y) must lie on the line $y = \frac{3}{4}x$.

Constraint : $\quad y = \dfrac{3}{4}x \qquad (2)$

Plugging this into the distance function (1) and simplifying we get:

$$\text{Distance} = \sqrt{(x-4)^2 + \left(\frac{3}{4}x - 0\right)^2}$$

$$f(x) = \sqrt{\frac{25}{16}x^2 - 8x + 16}$$

We want to find the absolute minimum of $f(x)$ on the **interval** $(-\infty, \infty)$. We choose this interval because (x, y) must be on the line $y = \frac{3}{4}x$ and the domain of this function is $(-\infty, \infty)$.

The absolute minimum of $f(x)$ will occur either at a critical point of $f(x)$ in $(0, \infty)$ or it will not exist because the interval is open. The critical points of $f(x)$ are solutions to $f'(x) = 0$.

$$f'(x) = 0$$

$$\left[\left(\frac{25}{16}x^2 - 8x + 16\right)^{1/2}\right]' = 0$$

$$\frac{1}{2}\left(\frac{25}{16}x^2 - 8x + 16\right)^{-1/2} \cdot \left(\frac{25}{16}x^2 - 8x + 16\right)' = 0$$

$$\frac{\frac{25}{8}x - 8}{2\sqrt{\frac{25}{16}x^2 - 8x + 16}} = 0$$

$$\frac{25}{8}x - 8 = 0$$

$$x = \frac{64}{25}$$

Plugging this into $f(x)$ we get:

$$f\left(\frac{64}{25}\right) = \sqrt{\frac{25}{16}\left(\frac{64}{25}\right)^2 - 8\left(\frac{64}{25}\right) + 16} = \frac{12}{5}$$

Taking the limits of $f(x)$ as x approaches the endpoints we get:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \sqrt{\frac{25}{16}x^2 - 8x + 16} = \infty$$
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \sqrt{\frac{25}{16}x^2 - 8x + 16} = \infty$$

both of which are larger than $\frac{12}{5}$. We conclude that the distance is an absolute minimum at $\boxed{x = \frac{64}{25}}$ and that the resulting distance is $\frac{12}{5}$. The last step is to find the corresponding value for y by plugging $x = \frac{64}{25}$ into equation (2).

$$\boxed{y = \frac{3}{4}\left(\frac{64}{25}\right) = \frac{48}{25}}$$