Math 180, Exam 2, Fall 2012
Problem 1 Solution

1. Find derivatives of the following functions:

(a) $f(x) = \tan^{-1}(\sqrt{x^2+1})$

(b) $f(x) = \ln\left(\dfrac{2^x}{x^2+1}\right)$

(c) $f(x) = x^{\sin(x)}$

Solution:

(a) The derivative is computed using the Chain Rule twice.

$$\begin{aligned} f'(x) &= \frac{1}{(\sqrt{x^2+1})^2+1} \cdot \frac{d}{dx}\sqrt{x^2+1} \\ &= \frac{1}{(x^2+1)+1} \cdot \frac{1}{2\sqrt{x^2+1}} \cdot \frac{d}{dx}(x^2+1) \\ &= \frac{1}{x^2+2} \cdot \frac{1}{2\sqrt{x^2+1}} \cdot 2x \end{aligned}$$

(b) We begin by rewriting the function using rules of logarithms.

$$\ln\left(\frac{2^x}{x^2+1}\right) = \ln(2^x) - \ln(x^2+1) = x\ln(2) - \ln(x^2+1)$$

Thus, the derivative is

$$\begin{aligned} f'(x) &= \frac{d}{dx}\, x\ln(2) - \frac{d}{dx}\ln(x^2+1) \\ &= \ln(2) - \frac{1}{x^2+1} \cdot \frac{d}{dx}(x^2+1) \\ &= \ln(2) - \frac{1}{x^2+1} \cdot 2x \end{aligned}$$

(c) To differentiate the function we rewrite it as

$$x^{\sin(x)} = e^{\ln x^{\sin(x)}} = e^{\sin(x)\ln(x)}$$

The derivative is then

$$\begin{aligned} f'(x) &= \frac{d}{dx} e^{\sin(x)\ln(x)} \\ &= e^{\sin(x)\ln(x)} \cdot \frac{d}{dx}\sin(x)\ln(x) \\ &= e^{\sin(x)\ln(x)} \cdot \left[\cos(x)\ln(x) + \sin(x)\cdot\frac{1}{x}\right] \\ &= x^{\sin(x)} \cdot \left[\cos(x)\ln(x) + \frac{\sin(x)}{x}\right] \end{aligned}$$

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Problem 2 Solution

2. Use implicit differentiation to find $\dfrac{dy}{dx}$ where $\tan(x^2y) = 4y$.

Solution: Taking the derivative of both sides of the equation we obtain

$$\begin{aligned}
\frac{d}{dx}\tan(x^2y) &= \frac{d}{dx}4y \\
\sec^2(x^2y)\cdot\frac{d}{dx}x^2y &= 2\frac{dy}{dx} \\
\sec^2(x^2y)\cdot\left(2xy + x^2\frac{dy}{dx}\right) &= 2\frac{dy}{dx} \\
2xy\sec^2(x^2y) + x^2\sec^2(x^2y)\frac{dy}{dx} &= 2\frac{dy}{dx}
\end{aligned}$$

Solving for $\dfrac{dy}{dx}$ we obtain

$$\begin{aligned}
\frac{dy}{dx}\left[2 - x^2\sec^2(x^2y)\right] &= 2xy\sec^2(x^2y) \\
\frac{dy}{dx} &= \frac{2xy\sec^2(x^2y)}{2 - x^2\sec^2(x^2y)}
\end{aligned}$$

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Problem 3 Solution

3. Consider the function $f(x) = 2x^3 + 3x^2 - 12x + 5$.

(a) Find the intervals on which f is increasing or decreasing.

(b) Find the points of local maximum and local minimum of f.

(c) Find the intervals on which f is concave up or concave down.

Solution: The first two derivatives of f are:

$$f'(x) = 6x^2 + 6x - 12, \quad f''(x) = 12x + 6$$

(a) To determine the intervals of monotonicity of f we begin by finding the points where $f' = 0$.

$$\begin{aligned} f'(x) &= 0 \\ 6x^2 + 6x - 12 &= 0 \\ 6(x^2 + x - 2) &= 0 \\ 6(x+2)(x-1) &= 0 \\ x = -2 \quad &\text{or} \quad x = 1 \end{aligned}$$

We summarize the desired information about f by way of the following table:

Interval	**Test #**, c	$f'(c)$	**Sign of** $f'(c)$	**Conclusion**
$(-\infty, -2)$	-3	24	$+$	increasing
$(-2, 1)$	0	-12	$-$	decreasing
$(1, \infty)$	2	24	$+$	increasing

Thus, f is increasing on $(-\infty, -2) \cup (1, \infty)$ and decreasing on $(-2, 1)$.

(b) By the First Derivative Test, f has

- a local maximum at $x = -2$ (f' changes sign from $+$ to $-$ across $x = -2$) and
- a local minimum at $x = 1$ (f' changes sign from $-$ to $+$ across $x = 1$).

(c) To determine the intervals of concavity of f we begin by finding the points where $f'' = 0$.

$$\begin{aligned} f''(x) &= 0 \\ 12x + 6 &= 0 \\ x &= -\frac{1}{2} \end{aligned}$$

We summarize the desired information about f by way of the following table:

Interval	**Test #**, c	$f''(c)$	**Sign of** $f''(c)$	**Conclusion**
$(-\infty, -\frac{1}{2})$	-1	-6	$-$	concave down
$(-\frac{1}{2}, \infty)$	0	6	$+$	concave up

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Problem 4 Solution

4. The product of two positive numbers is 200. Find the two numbers if the sum of their squares is to be as small as possible.

Solution: Let x and y be the numbers of interest. Since their product is known to be 200 we have

$$xy = 200$$

The function to be minimized is the sum of their squares, i.e. $x^2 + y^2$. Solving the constraint equation for y yields:

$$y = \frac{200}{x}$$

The function is then

$$f(x) = x^2 + \left(\frac{200}{x}\right)^2 = x^2 + \frac{200^2}{x^2}$$

Note that the domain of f is $(0, \infty)$ since x must be positive.

The rest of our analysis begins with finding the critical points of f.

$$\begin{aligned} f'(x) &= 0 \\ 2x - \frac{2 \cdot 200^2}{x^3} &= 0 \\ 2x^4 - 2 \cdot 200^2 &= 0 \\ x^4 &= 200^2 \\ x^2 &= 200 \\ x &= 10\sqrt{2} \end{aligned}$$

Since f is concave up on its domain ($f'' = 2 + 6 \cdot 200^2/x^4 > 0$ for all $x > 0$), we know that f has an absolute minimum at $x = 10\sqrt{2}$. The corresponding y-value is

$$y = \frac{200}{10\sqrt{2}} = 10\sqrt{2}$$

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Problem 5 Solution

5. Let f be a function such that $f(3) = 1$ and $f'(3) = 2$.

(a) Use the linear approximation of f about $x = 3$ to estimate $f(3.1)$.

(b) Let g be the inverse of f. Find $g(1)$ and $g'(1)$.

Solution:

(a) The linear approximation of f about $x = 3$ is

$$L(x) = f(3) + f'(3)(x - 3) = 1 + 2(x - 3)$$

The approximate value of $f(3.1)$ is then

$$f(3.1) \approx L(3.1) = 1 + 2(3.1 - 3) = 1.2$$

(b) By the property of inverses, we know that if $f(a) = b$ then $g(b) = a$ if g is the inverse of f. Thus, since $f(3) = 1$ we know that $g(1) = 3$. Furthermore, the derivative $g'(b)$ has the formula

$$g'(b) = \frac{1}{f'(g(b))}$$

Therefore, the derivative $g'(1)$ is

$$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(3)} = \frac{1}{2}$$