Math 180, Exam 2, Fall 2013
Problem 1 Solution

1. Compute each derivative below.

   (a) $\dfrac{d}{dx}\left(2^x \cdot \arcsin(x^3)\right)$ [Note: $\arcsin(x) = \sin^{-1}(x)$]

   (b) $\dfrac{d}{dx}(\sin(x))^x$ [Hint: Logarithmic differentiation may be useful.]

Solution:

(a) Using the Product Rule and the derivative rules for $\arcsin(x)$ and b^x we have

$$\frac{d}{dx}\left(2^x \cdot \arcsin(x^3)\right) = 2^x \cdot \frac{d}{dx}\arcsin(x^3) + \arcsin(x^3) \cdot \frac{d}{dx}2^x$$
$$\frac{d}{dx}\left(2^x \cdot \arcsin(x^3)\right) = 2^x \cdot \frac{1}{\sqrt{1-(x^3)^2}} \cdot \frac{d}{dx}x^3 + \arcsin(x^3) \cdot \ln(2) \cdot 2^x$$
$$\boxed{\frac{d}{dx}\left(2^x \cdot \arcsin(x^3)\right) = 2^x \cdot \frac{1}{\sqrt{1-(x^3)^2}} \cdot 3x^2 + \arcsin(x^3) \cdot \ln(2) \cdot 2^x}$$

(b) Let $y = (\sin(x))^x$. Then $\ln(y) = \ln(\sin(x))^x = x \cdot \ln(\sin(x))$. Differentiating both sides of this equation with respect to x yields

$$\frac{d}{dx}[\ln(y)] = \frac{d}{dx}[x \cdot \ln(\sin(x))]$$
$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{d}{dx}\ln(\sin(x)) + \ln(\sin(x)) \cdot \frac{d}{dx}x$$
$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{\sin(x)} \cdot \frac{d}{dx}\sin(x) + \ln(\sin(x)) \cdot 1$$
$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{\sin(x)} \cdot \cos(x) + \ln(\sin(x))$$
$$\frac{dy}{dx} = y\left(x \cdot \frac{\cos(x)}{\sin(x)} + \ln(\sin(x))\right)$$
$$\boxed{\frac{dy}{dx} = (\sin(x))^x\left(x \cdot \frac{\cos(x)}{\sin(x)} + \ln(\sin(x))\right)}$$

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Problem 2 Solution

2. Consider the function $g(x) = x\sqrt{x+1}$.

(a) State the domain of g.

(b) State the intervals where g is increasing and those where g is decreasing.

Solution: The domain of g is $x \geq -1$. **Solution**: We begin by finding the critical points of g. The derivative $g'(x)$ is found using the Product Rule:

$$\begin{aligned}
g'(x) &= x \cdot \frac{d}{dx}\sqrt{x+1} + \sqrt{x+1} \cdot \frac{d}{dx}x \\
g'(x) &= x \cdot \frac{1}{2\sqrt{x+1}} \cdot \frac{d}{dx}(x+1) + \sqrt{x+1} \cdot 1 \\
g'(x) &= x \cdot \frac{1}{2\sqrt{x+1}} \cdot 1 + \sqrt{x+1} \\
g'(x) &= \frac{x}{2\sqrt{x+1}} + \sqrt{x+1}
\end{aligned}$$

The critical points of g are solutions to $g'(x) = 0$:

$$\begin{aligned}
g'(x) &= 0 \\
\frac{x}{2\sqrt{x+1}} + \sqrt{x+1} &= 0 \\
x + 2\sqrt{x+1} \cdot \sqrt{x+1} &= 0 \\
x + 2(x+1) &= 0 \\
3x + 2 &= 0 \\
x &= -\frac{2}{3}
\end{aligned}$$

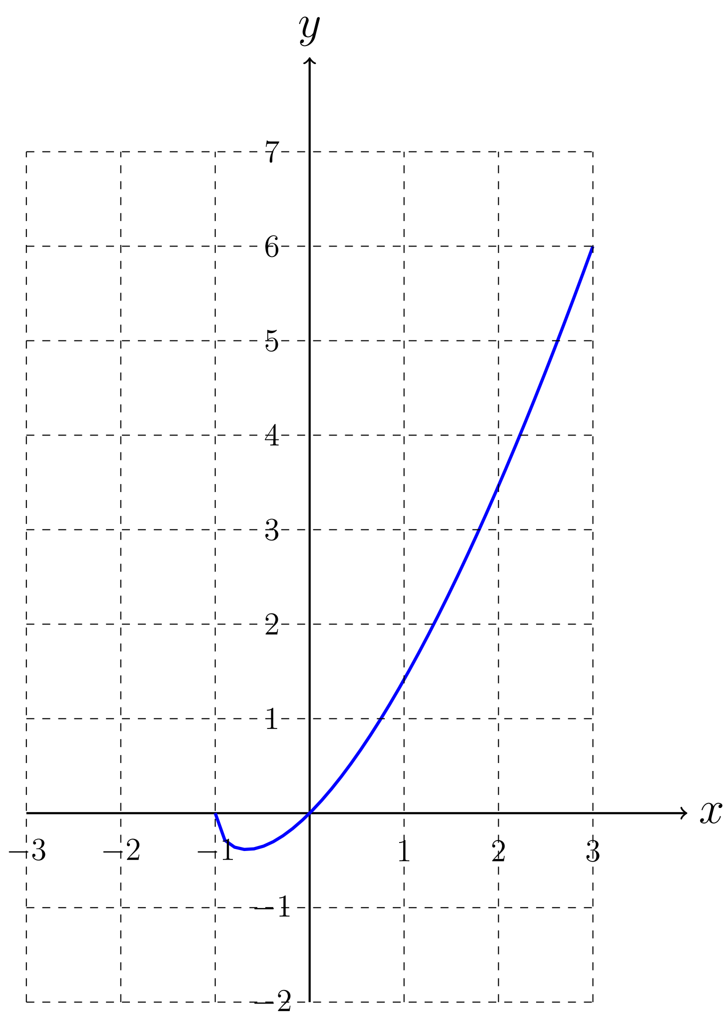
Summarize your results in the table below. **Note: It may not be necessary to use every row in the table.**

Interval	**Test #, c**	$f'(c)$	**Sign of $f'(c)$**	**Conclusion**
$(-1, -2/3)$	$-3/4$	$-1/4$	$-$	decreasing
$(-2/3, \infty)$	0	1	$+$	increasing

Find all local extrema of g, or state that none exist.

Solution: $g(-2/3)$ is a local minimum of g because the derivative of g changes from negative to positive across $x = -2/3$. There is no local maximum.

Using the information in parts (a)-(c), sketch g.



Consider the function $f(x) = \sqrt[3]{x}$. Find the best linear approximation to f at $x = 8$.

Solution: The linear approximation is given by

$$L(x) = f(8) + f'(8)(x - 8)$$

The derivative $f'(x)$ is

$$f'(x) = \frac{d}{dx} x^{1/3} = \frac{1}{3} x^{-2/3}$$

At $x = 8$ we have

$$f'(8) = \frac{1}{3} 8^{-2/3} = \frac{1}{12}, \quad f(8) = 8^{1/3} = 2$$

. Therefore, the linear approximation is

$$\boxed{L(x) = 2 + \frac{1}{12}(x - 8)}$$

Use part (a) to estimate $\sqrt[3]{7.7}$. Write your answer in the form $\frac{a}{b}$ where a and b are integers.

Solution: An estimate for $\sqrt[3]{7.7}$ is

$$\sqrt[3]{7.7} \approx L(7.7) = 2 + \frac{1}{12}(7.7 - 8)$$
$$\sqrt[3]{7.7} = 2 + \frac{1}{12} \cdot (-0.3)$$
$$\sqrt[3]{7.7} = 2 + \frac{1}{12}\left(-\frac{3}{10}\right)$$
$$\sqrt[3]{7.7} = 2 - \frac{1}{40}$$
$$\boxed{\sqrt[3]{7.7} = \frac{79}{40}}$$

Is your answer in part (b) an underestimate or an overestimate? Justify your answer.

Solution: To determine whether the estimate is an overestimate or an underestimate, we determine whether f is concave up or concave down at $x = 8$. The second derivative $f''(x)$ is

$$f''(x) = \frac{d}{dx} \frac{1}{3} x^{-2/3} = -\frac{2}{9} x^{-5/3}$$

At $x = 8$ we have

$$f''(8) = -\frac{2}{9} 8^{-5/3} < 0$$

Therefore, f is concave down at $x = 8$ and the tangent line lies above the graph of f near $x = 8$. Thus, the estimate in part (b) is an **overestimate**.

Consider the line $y = -\frac{7}{3}x + 10$. Find the x-coordinate of the point (x, y) on the given line such that the sum of the squares of both coordinates (i.e. $x^2 + y^2$) is minimized.

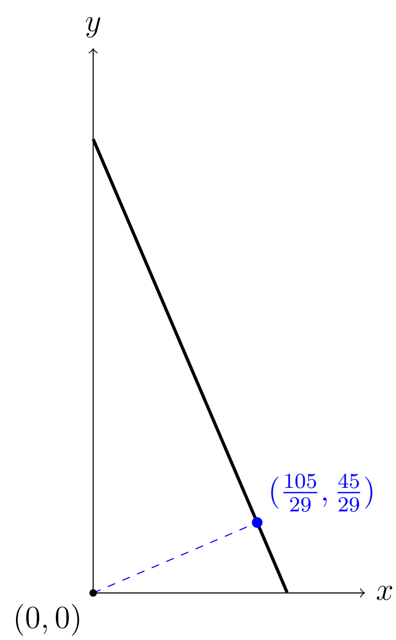
Solution: We want to minimize the function $x^2 + y^2$ given the constraint $y = -\frac{7}{3}x + 10$. Thus, we have

$$f(x) = x^2 + \left(-\frac{7}{3}x + 10\right)^2$$
$$f(x) = x^2 + \frac{49}{9}x^2 - \frac{140}{3}x + 100$$
$$\boxed{f(x) = \frac{58}{9}x^2 - \frac{140}{3}x + 100}$$

The critical points of f are solutions to $f'(x) = 0$:

$$f'(x) = 0$$
$$\frac{116}{9}x - \frac{140}{3} = 0$$
$$\frac{116}{9}x = \frac{140}{3}$$
$$\boxed{x = \frac{105}{29}}$$

The second derivative of f is $f''(x) = \frac{116}{9} > 0$. Thus, f is concave up for all x and $x = \frac{105}{29}$ must correspond to an absolute minimum of f.



Let $f(x) = xe^{4x}$ and consider the interval $[-3, 0]$. Find the absolute maximum and absolute minimum values of the function on the interval.

Solution: To find the absolute extrema we evaluate f at the critical points in $(-3, 0)$ and at the endpoints $-3, 0$. The critical points of f are solutions to $f'(x) = 0$:

$$\begin{aligned} f'(x) &= 0 \\ x \cdot \frac{d}{dx}e^{4x} + e^{4x} \cdot \frac{d}{dx}x &= 0 \\ x \cdot 4e^{4x} + e^{4x} \cdot 1 &= 0 \\ e^{4x}(4x+1) &= 0 \\ 4x + 1 &= 0 \\ x &= -\frac{1}{4} \end{aligned}$$

The value of f at $x = -3, -\frac{1}{4}, 0$ are:

$$f(-3) = -3e^{-12}, \quad f(-1/4) = (-1/4)e^{-1}, \quad f(0) = 0$$

The largest of the above values is 0 (absolute maximum) and the smallest is $(-1/4)e^{-1}$ (absolute minimum).