Math 180, Exam 2, Practice Fall 2009
Problem 1 Solution

1. Differentiate the functions: (do not simplify)

$$f(x) = x\ln(x^2+1), \qquad f(x) = xe^{\sqrt{x}}$$

$$f(x) = \arcsin(2x+1) = \sin^{-1}(3x+1), \qquad f(x) = \frac{e^{3x}}{\ln x}$$

Solution: For the first function, we use the Product and Chain Rules.

$$\begin{aligned} f'(x) &= \left[x\ln(x^2+1)\right]' \\ &= x[\ln(x^2+1)]' + (x)'\ln(x^2+1) \\ &= x\cdot\frac{1}{x^2+1}\cdot(x^2+1)' + 1\cdot\ln(x^2+1) \\ &= x\cdot\frac{1}{x^2+1}\cdot 2x + \ln(x^2+1) \\ &= \boxed{\frac{2x^2}{x^2+1} + \ln(x^2+1)} \end{aligned}$$

For the second function, we use the Product and Chain Rules.

$$\begin{aligned} f'(x) &= \left(xe^{\sqrt{x}}\right)' \\ &= x\left(e^{\sqrt{x}}\right)' + (x)'e^{\sqrt{x}} \\ &= x\cdot e^{\sqrt{x}}\cdot\left(\sqrt{x}\right)' + 1\cdot e^{\sqrt{x}} \\ &= x\cdot e^{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}} + e^{\sqrt{x}} \\ &= \boxed{\frac{1}{2}\sqrt{x}e^{\sqrt{x}} + e^{\sqrt{x}}} \end{aligned}$$

For the third function, we use the Chain Rule.

$$\begin{aligned} f'(x) &= \left[\sin^{-1}(3x+1)\right]' \\ &= \frac{1}{\sqrt{1-(3x+1)^2}}\cdot(3x+1)' \\ &= \boxed{\frac{1}{\sqrt{1-(3x+1)^2}}\cdot 3} \end{aligned}$$

For the fourth function, we use the Quotient and Chain Rules.

$$\begin{aligned}
f'(x) &= \left(\frac{e^{3x}}{\ln x}\right)' \\
&= \frac{(\ln x)(e^{3x})' - (e^{3x})(\ln x)'}{(\ln x)^2} \\
&= \frac{(\ln x)(e^{3x})(3x)' - (e^{3x})(\frac{1}{x})}{(\ln x)^2} \\
&= \boxed{\frac{3e^{3x}\ln x - e^{3x}\cdot\frac{1}{x}}{(\ln x)^2}}
\end{aligned}$$

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Problem 2 Solution

2. The following table of values is provided for the functions f, g, and their derivatives:

x	1	3
$f(x)$	2	4
$f'(x)$	1	5
$g(x)$	3	-2
$g'(x)$	2	-3

Let $h(x) = f(g(x))$ and compute $h'(1)$.

Solution: Using the Chain Rule, the derivative of $h(x)$ is:

$$h'(x) = f'(g(x))g'(x)$$

At $x = 1$ we have:

$$\begin{aligned} h'(1) &= f'(g(1))g'(1) \\ &= f'(3)g'(1) \\ &= (5)(2) \\ &= \boxed{10} \end{aligned}$$

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Problem 3 Solution

3. Differentiate the following functions: (do not simplify)

$$f(x) = \sin(x^2 + 5x + 2), \qquad f(x) = \ln(x + \cos x), \qquad f(x) = (1 + \ln x)^{3/4}$$

Solution: For the first function, we use the Chain Rule.

$$\begin{aligned} f'(x) &= \left[\sin(x^2 + 5x + 2)\right]' \\ &= \cos(x^2 + 5x + 2) \cdot (x^2 + 5x + 2)' \\ &= \boxed{\cos(x^2 + 5x + 2) \cdot (2x + 5)} \end{aligned}$$

For the second function, we use the Chain Rule.

$$\begin{aligned} f'(x) &= [\ln(x + \cos x)]' \\ &= \frac{1}{x + \cos x} \cdot (x + \cos x)' \\ &= \boxed{\frac{1}{x + \cos x} \cdot (1 - \sin x)} \end{aligned}$$

For the third function, we use the Chain Rule.

$$\begin{aligned} f'(x) &= \left[(1 + \ln x)^{3/4}\right]' \\ &= \frac{3}{4}(1 + \ln x)^{-1/4} \cdot (1 + \ln x)' \\ &= \boxed{\frac{3}{4}(1 + \ln x)^{-1/4} \cdot \frac{1}{x}} \end{aligned}$$

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Problem 4 Solution

4. Find the derivative of the function $y = x^x$.

Solution: To find the derivative we use logarithmic differentiation. We start by taking the natural logarithm of both sides of the equation.

$$\begin{aligned} y &= x^x \\ \ln y &= \ln x^x \\ \ln y &= x \ln x \end{aligned}$$

Then we implicitly differentiate the equation and solve for y'.

$$\begin{aligned} (\ln y)' &= (x \ln x)' \\ \frac{1}{y} \cdot y' &= x(\ln x)' + (\ln x)(x)' \\ \frac{1}{y} \cdot y' &= x \cdot \frac{1}{x} + \ln x \cdot 1 \\ \frac{1}{y} \cdot y' &= 1 + \ln x \\ y' &= y(1 + \ln x) \\ &\boxed{y' = x^x(1 + \ln x)} \end{aligned}$$

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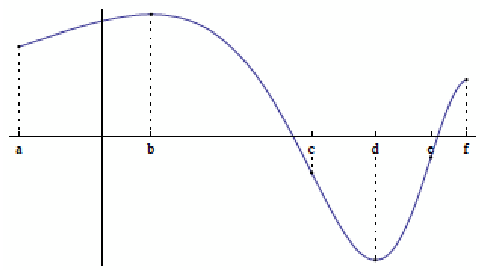
Problem 5 Solution

5. The graph of a function $f(x)$ is given below. List the intervals on which f is increasing, decreasing, concave up, and concave down.



Solution: $f(x)$ is increasing on $(a, b) \cup (d, f)$ because $f'(x) > 0$ for these values of x. $f(x)$ is decreasing on (b, d) because $f'(x) < 0$ for these values of x. $f(x)$ is concave up on (c, e) because $f'(x)$ is increasing for these values of x. $f(x)$ is concave down on $(a, c) \cup (e, f)$ because $f'(x)$ is decreasing for these value of x.

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Problem 6 Solution

6. Find the equation of the tangent to the curve $y^2x + x + 2y = 4$ at the point $(1, 1)$.

Solution: We find y' using implicit differentiation.

$$\begin{aligned}
y^2x + x + 2y &= 4\\
(y^2x)' + (x)' + (2y)' &= (4)'\\
\left[(y^2)(x)' + (x)(y^2)'\right] + 1 + 2y' &= 0\\
\left[(y^2)(1) + (x)(2yy')\right] + 1 + 2y' &= 0\\
y^2 + 2xyy' + 1 + 2y' &= 0\\
2xyy' + 2y' &= -y^2 - 1\\
y'(2xy + 2) &= -y^2 - 1\\
y' &= \frac{-y^2 - 1}{2xy + 2}
\end{aligned}$$

At the point $(1, 1)$, the value of y' is:

$$y'(1, 1) = \frac{-1^2 - 1}{2(1)(1) + 2} = -\frac{1}{2}$$

This represents the slope of the tangent line. An equation for the tangent line is then:

$$\boxed{y - 1 = -\frac{1}{2}(x - 1)}$$

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Problem 7 Solution

7. Let $f(x) = xe^x$.

(a) Find and classify the critical points of f.

(b) Is there a global minimum of f over the entire real line? Why or why not?

Solution:

(a) The critical points of $f(x)$ are the values of x for which either $f'(x) = 0$ or $f'(x)$ does not exist. Since $f(x)$ is a product of two infinitely differentiable functions, we know that $f'(x)$ exists for all $x \in \mathbb{R}$. Therefore, the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ (xe^x)' &= 0 \\ (x)(e^x)' + (e^x)(x)' &= 0 \\ xe^x + e^x &= 0 \\ e^x(x+1) &= 0 \\ x &= -1 \end{aligned}$$

$\boxed{x = -1}$ is the only critical point because $e^x > 0$ for all $x \in \mathbb{R}$.

We use the First Derivative Test to classify the critical point $x = -1$. The domain of f is $(-\infty, \infty)$. Therefore, we divide the domain into the two intervals $(-\infty, -1)$ and $(-1, \infty)$. We then evaluate $f'(x)$ at a test point in each interval to determine where $f'(x)$ is positive and negative.

Interval	**Test Number,** c	$f'(c)$	**Sign of** $f'(c)$
$(-\infty, -1)$	-2	$-e^{-2}$	$-$
$(-1, \infty)$	0	1	$+$

Since f changes sign from $-$ to $+$ at $x = -1$ the First Derivative Test implies that $f(-1) = -e^{-1}$ is a **local minimum**.

(b) From the table in part (a), we conclude that f is decreasing on the interval $(-\infty, -1)$ and increasing on the interval $(-1, \infty)$. Therefore, $\boxed{f(-1) = -e^{-1}}$ is the global minimum of f over the entire real line.

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Problem 8 Solution

8. Find the minimum and the maximum values of the function $f(x) = x^3 - 3x$ over the interval $[0, 2]$.

Solution: The minimum and maximum values of $f(x)$ will occur at a critical point in the interval $[0, 2]$ or at one of the endpoints. The critical points are the values of x for which either $f'(x) = 0$ or $f'(x)$ does not exist. Since $f(x)$ is a polynomial, $f'(x)$ exists for all $x \in \mathbb{R}$. Therefore, the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(x^3 - 3x\right)' &= 0 \\ 3x^2 - 3 &= 0 \\ 3(x^2 - 1) &= 0 \\ 3(x+1)(x-1) &= 0 \\ x = -1,\ x &= 1 \end{aligned}$$

The critical point $x = -1$ lies outside $[0, 2]$ but the critical point $x = 1$ is in $[0, 2]$. Therefore, we check the value of $f(x)$ at $x = 0,\ 1$, and 2.

$$\begin{aligned} f(0) &= 0^3 - 3(0) = 0 \\ f(1) &= 1^3 - 3(1) = -2 \\ f(2) &= 2^3 - 3(2) = 2 \end{aligned}$$

The minimum value of $f(x)$ on $[0, 2]$ is $\boxed{-2}$ because it is the smallest of the above values of f. The maximum is $\boxed{2}$ because it is the largest.

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Problem 9 Solution

9. A function f is defined on $[0, 2]$ by $f(x) = x^2 + x + 1$ for $0 \leq x \leq 2$. Let g be the inverse function of f. Find $g'(3)$.

Solution: The value of $g'(3)$ is given by the formula:

$$g'(3) = \frac{1}{f'(g(3))}$$

It isn't necessary to find a formula for $g(x)$ to find $g(3)$. We will use the fact that $f(1) = 1^2 + 1 + 1 = 3$ to say that $g(3) = 1$ by the property of inverses. The derivative of $f(x)$ is $f'(x) = 2x + 1$. Therefore,

$$\begin{aligned} g'(3) &= \frac{1}{f'(g(3))} \\ &= \frac{1}{f'(1)} \\ &= \frac{1}{2(1) + 1} \\ &= \boxed{\frac{1}{3}} \end{aligned}$$

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Problem 10 Solution

10. Find the limits

(a) $\lim_{x \to 0} \frac{1 - \cos(3x)}{x^2}$ (b) $\lim_{x \to \pi/6} \frac{1 - \cos(3x)}{x^2}$

Solution:

(a) Upon substituting $x = 0$ into the function $\frac{1-\cos(3x)}{x^2}$ we get

$$\frac{1 - \cos(3(0))}{0^2} = \frac{0}{0}$$

which is indeterminate. We resolve the indeterminacy using L'Hôpital's Rule.

$$\begin{aligned}\lim_{x \to 0} \frac{1 - \cos(3x)}{x^2} &\stackrel{\text{L'H}}{=} \lim_{x \to 0} \frac{(1 - \cos(3x))'}{(x^2)'} \\ &= \lim_{x \to 0} \frac{3\sin(3x)}{2x}\end{aligned}$$

Upon substituting $x = 0$ into $\frac{3\sin(3x)}{2x}$ we get

$$\frac{3\sin(3(0))}{2(0)} = \frac{0}{0}$$

which is indeterminate. We resolve this indeterminacy using another application of L'Hôpital's Rule.

$$\begin{aligned}\lim_{x \to 0} \frac{1 - \cos(3x)}{x^2} &= \lim_{x \to 0} \frac{3\sin(3x)}{2x} \\ &\stackrel{\text{L'H}}{=} \frac{(3\sin(3x))'}{(2x)'} \\ &= \lim_{x \to 0} \frac{9\cos(3x)}{2} \\ &= \boxed{\frac{9}{2}}\end{aligned}$$

(b) Upon substituting $x = \frac{\pi}{6}$ into the function $\frac{1-\cos(3x)}{x^2}$ we get

$$\frac{1 - \cos(3(\frac{\pi}{6}))}{(\frac{\pi}{6})^2} = \frac{1 - 0}{(\frac{\pi}{6})^2} = \frac{36}{\pi^2}$$

Therefore, the value of the limit is:

$$\lim_{x \to \pi/6} \frac{1 - \cos(3x)}{x^2} = \boxed{\frac{36}{\pi^2}}$$

Substitution works in this problem because the function is continuous at $x = \frac{\pi}{6}$.

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Problem 11 Solution

11. Find $\lim_{x \to 0^+} x \ln x$.

Solution: As $x \to 0^+$ we find that $x \ln x \to 0 \cdot (-\infty)$ which is indeterminate. However, it is not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ which is required to use L'Hôpital's Rule. To get the limit into one of the two required forms, we rewrite $x \ln x$ as follows:

$$x \ln x = \frac{\ln x}{\frac{1}{x}}$$

As $x \to 0^+$, we find that $\frac{\ln x}{1/x} \to \frac{-\infty}{\infty}$. We can now use L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x \to 0^+} x \ln x &= \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} \\
&\overset{\text{L'H}}{=} \lim_{x \to 0^+} \frac{(\ln x)'}{(\frac{1}{x})'} \\
&= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\
&= \lim_{x \to 0^+} -x \\
&= \boxed{0}
\end{aligned}$$

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Problem 12 Solution

12. Find the critical points of the function $f(x) = x^3 + x^2 - x + 5$ and determine if they correspond to local maxima, minima, or neither.

Solution: The critical points of $f(x)$ are the values of x for which either $f'(x)$ does not exist or $f'(x) = 0$. Since $f(x)$ is a polynomial, $f'(x)$ exists for all $x \in \mathbb{R}$ so the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(x^3 + x^2 - x + 5\right)' &= 0 \\ 3x^2 + 2x - 1 &= 0 \\ (3x-1)(x+1) &= 0 \\ x = \frac{1}{3},\ x &= -1 \end{aligned}$$

Thus, $\boxed{x = -1}$ and $\boxed{x = \frac{1}{3}}$ are the critical points of f. We will use the Second Derivative Test to classify the points as either local maxima or a local minima. The second derivative is $f''(x) = 6x + 2$. The values of $f''(x)$ at the critical points are:

$$\begin{aligned} f''(-1) &= 6(-1) + 2 = -4 \\ f''\left(\frac{1}{3}\right) &= 6\left(\frac{1}{3}\right) + 2 = 4 \end{aligned}$$

Since $f''(-1) < 0$ the Second Derivative Test implies that $f(-1) = 6$ is a local maximum and since $f''(\frac{1}{3}) > 0$ the Second Derivative Test implies that $f(\frac{1}{3}) = \frac{130}{27}$ is a local minimum.

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Problem 13 Solution

13. Let $f(x) = x^4 + 2x^2$. Determine the intervals on which f is increasing or decreasing and on which f is concave up or down.

Solution: We begin by finding the critical points of $f(x)$. These occur when either $f'(x)$ does not exist or $f'(x) = 0$. Since $f(x)$ is a polynomial we know that $f'(x)$ exists for all $x \in \mathbb{R}$. Therefore, the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(x^4 + 2x^2\right)' &= 0 \\ 4x^3 + 4x &= 0 \\ 4x(x^2 + 1) &= 0 \\ x &= 0 \end{aligned}$$

The domain of $f(x)$ is $(-\infty, \infty)$. We now split the domain into the two intervals $(-\infty, 0)$ and $(0, \infty)$. We then evaluate $f'(x)$ at a test point in each interval to determine the intervals of monotonicity.

Interval	**Test Point**, c	$f'(c)$	**Sign of** $f'(c)$
$(-\infty, 0)$	-1	$f'(-1) = -8$	$-$
$(0, \infty)$	1	$f'(1) = 8$	$+$

Using the table we conclude that $f(x)$ is increasing on $(0, \infty)$ because $f'(x) > 0$ for all $x \in (0, \infty)$ and $f(x)$ is decreasing on $(-\infty, 0)$ because $f'(x) < 0$ for all $x \in (-\infty, 0)$.

To find the intervals of concavity we begin by finding solutions to $f''(x) = 0$.

$$\begin{aligned} f''(x) &= 0 \\ \left(4x^4 + 4x\right)' &= 0 \\ 12x^2 + 4 &= 0 \end{aligned}$$

This equations has no solutions. In fact, $f''(x) = 12x^2 + 4 > 0$ for all $x \in \mathbb{R}$. Therefore, the function $f(x)$ is concave up on $(-\infty, \infty)$.

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Problem 14 Solution

14. Let $f(x) = 2x^3 + 3x^2 - 12x + 1$.

(a) Find the critical points of f.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

(c) Find the local minima and maxima of f. Compute x and $f(x)$ for each local extremum.

(d) Determine the intervals on which f is concave up and the intervals on which f is concave down.

(e) Find the points of inflection of f.

(f) Sketch the graph of f.

Solution:

(a) The critical points of $f(x)$ are the values of x for which either $f'(x)$ does not exist or $f'(x) = 0$. Since $f(x)$ is a polynomial, $f'(x)$ exists for all $x \in \mathbb{R}$ so the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(2x^3 + 3x^2 - 12x + 1\right)' &= 0 \\ 6x^2 + 6x - 12 &= 0 \\ 6(x^2 + x - 2) &= 0 \\ 6(x+2)(x-1) &= 0 \\ x = -2,\ x &= 1 \end{aligned}$$

Thus, $\boxed{x = -2}$ and $\boxed{x = 1}$ are the critical points of f.

(b) The domain of f is $(-\infty, \infty)$. We now split the domain into the three intervals $(-\infty, -2)$, $(-2, 1)$, and $(1, \infty)$. We then evaluate $f'(x)$ at a test point in each interval to determine the intervals of monotonicity.

Interval	**Test Point**, c	$f'(c)$	**Sign of** $f'(c)$
$(-\infty, -2)$	-3	$f'(-3) = 24$	$+$
$(-2, 1)$	0	$f'(0) = -12$	$-$
$(1, \infty)$	2	$f'(2) = 24$	$+$

Using the table, we conclude that f is increasing on $(-\infty,-2)\cup(1,\infty)$ because $f'(x) > 0$ for all $x \in (-\infty,-2)\cup(1,\infty)$ and f is decreasing on $(-2,1)$ because $f'(x) < 0$ for all $x \in (-2,1)$.

(c) Since f' changes sign from $+$ to $-$ at $x=-2$ the First Derivative Test implies that $\boxed{f(-2)=21}$ is a local maximum and since f' changes sign from $-$ to $+$ at $x=1$ the First Derivative Test implies that $\boxed{f(1)=-6}$ is a local minimum.

(d) To determine the intervals of concavity we start by finding solutions to the equation $f''(x)=0$ and where $f''(x)$ does not exist. However, since $f(x)$ is a polynomial we know that $f''(x)$ will exist for all $x \in \mathbb{R}$. The solutions to $f''(x)=0$ are:

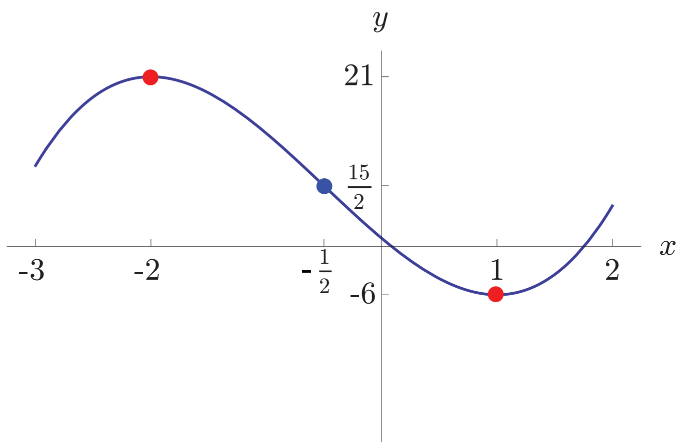
$$\begin{aligned} f''(x) &= 0 \\ \left(6x^2+6x-12\right)' &= 0 \\ 12x+6 &= 0 \\ x &= -\frac{1}{2} \end{aligned}$$

We now split the domain into the two intervals $(-\infty,-\frac{1}{2})$ and $(-\frac{1}{2},\infty)$. We then evaluate $f''(x)$ at a test point in each interval to determine the intervals of concavity.

Interval	**Test Point**, c	$f'(c)$	**Sign of** $f'(c)$
$(-\infty,-\frac{1}{2})$	-1	$f''(-1)=-6$	$-$
$(-\frac{1}{2},\infty)$	0	$f''(0)=6$	$+$

Using the table, we conclude that f is concave down on $(-\infty,-\frac{1}{2})$ because $f''(x) < 0$ for all $x \in (-\infty,-\frac{1}{2})$ and f is concave up on $(-\frac{1}{2},\infty)$ because $f''(x) > 0$ for all $x \in (-\frac{1}{2},\infty)$.

(e) The inflection points of $f(x)$ are the points where $f''(x)$ changes sign. We can see in the above table that $f''(x)$ changes sign at $x=-\frac{1}{2}$. Therefore, $\boxed{x=-\frac{1}{2}}$ is an inflection point.

(f) 

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Problem 15 Solution

15. Use the Newton approximation method to estimate the positive root of the equation $x^2 - 2 = 0$. Begin with $x_0 = 2$ and compute x_1. Present your answer as a fraction with integer numerator and denominator.

Solution: The Newton's method formula to compute x_1 is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

where $f(x) = x^2 - 2$. The derivative $f'(x)$ is $f'(x) = 2x$. Plugging $x_0 = 2$ into the formula we get:

$$x_1 = x_0 - \frac{x_0^2 - 2}{2x_0}$$
$$x_1 = 2 - \frac{2^2 - 2}{2(2)}$$
$$x_1 = 2 - \frac{2}{4}$$
$$\boxed{x_1 = \frac{3}{2}}$$

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Problem 16 Solution

16. A rectangle has its left lower corner at $(0,0)$ and its upper right corner on the graph of

$$f(x) = x^2 + \frac{1}{x^2}$$

(a) Express its area as a function of x.

(b) For which x is the area minimum and what is this area?

Solution:

(a) The dimensions of the rectangle are x and y. Therefore, the area of the rectangle has the equation:

$$\text{Area} = xy \tag{1}$$

We are asked to write the area as a function of x alone. Therefore, we must find an equation that relates x to y so that we can eliminate y from the area equation. This equation is

$$y = x^2 + \frac{1}{x^2} \tag{2}$$

because (x, y) must lie on this curve. Plugging this into the area equation we get:

$$\text{Area} = x\left(x^2 + \frac{1}{x^2}\right)$$

$$\boxed{g(x) = x^3 + \frac{1}{x}}$$

(b) We seek the value of x that minimizes $g(x)$. The interval in the problem is $(0, \infty)$ because the domain of $f(x)$ is $(-\infty, 0) \cup (0, \infty)$ but (x, y) must be in the first quadrant.

The absolute minimum of $f(x)$ will occur either at a critical point of $f(x)$ in $(0, \infty)$ or it will not exist because the interval is open. The critical points of $f(x)$ are solutions to $f'(x) = 0$.

$$\begin{aligned}
f'(x) &= 0 \\
\left(x^3 + \frac{1}{x}\right)' &= 0 \\
3x^2 - \frac{1}{x^2} &= 0 \\
3x^4 - 1 &= 0 \\
x &= \pm\frac{1}{\sqrt[4]{3}}
\end{aligned}$$

However, since $x = -\frac{1}{\sqrt[4]{3}}$ is outside $(0, \infty)$, the only critical point is $x = \frac{1}{\sqrt[4]{3}}$. Plugging this into $g(x)$ we get:

$$f\left(\frac{1}{\sqrt[4]{3}}\right) = \left(\frac{1}{\sqrt[4]{3}}\right)^3 + \frac{1}{\frac{1}{\sqrt[4]{3}}} = \frac{1}{\sqrt[4]{27}} + \sqrt[4]{3}$$

Taking the limits of $f(x)$ as x approaches the endpoints we get:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left(x^3 + \frac{1}{x}\right) = 0 + \infty = \infty$$
$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} \left(x^3 + \frac{1}{x}\right) = \infty + 0 = \infty$$

both of which are larger than $\frac{1}{\sqrt[4]{27}} + \sqrt[4]{3}$. We conclude that the area is an absolute minimum at $\boxed{x = \frac{1}{\sqrt[4]{3}}}$ and that the resulting area is $\boxed{\frac{1}{\sqrt[4]{27}} + \sqrt[4]{3}}$.

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Problem 17 Solution

17. A box has square base and total surface area equal to 12 m^2.

(a) Express its volume as a function of x, the length of one side of the base.

(b) Find the maximum volume of such a box.

Solution:

(a) We begin by letting x be the length of one side of the base and y be the height of the box. The volume then has the equation:

$$\text{Volume} = x^2 y \tag{1}$$

We are asked to write the volume as a function of width, x. Therefore, we must find an equation that relates x to y so that we can eliminate y from the volume equation.

The constraint in the problem is that the total surface area is 12. This gives us the equation

$$2x^2 + 4xy = 12 \tag{2}$$

Solving this equation for y we get

$$\begin{aligned} 2x^2 + 4xy &= 12 \\ x^2 + 2xy &= 6 \\ y &= \frac{6 - x^2}{2x} \end{aligned} \tag{3}$$

We then plug this into the volume equation (1) to write the volume in terms of x only.

$$\begin{aligned} \text{Volume} &= x^2 y \\ \text{Volume} &= x^2 \left(\frac{6 - x^2}{2x} \right) \\ & \boxed{f(x) = 3x - \frac{1}{2}x^3} \end{aligned} \tag{4}$$

(b) We seek the value of x that maximizes $f(x)$. The interval in the problem is $(0, \sqrt{6}]$. We know that $x > 0$ because x must be positive and nonzero (otherwise, the surface area would be 0 and it must be 12). It is possible that $y = 0$ in which case the surface area constraint would give us $2x^2 + 4x(0) = 12 \quad \Rightarrow \quad x^2 = 6 \quad \Rightarrow \quad x = \sqrt{6}$.

The absolute maximum of $f(x)$ will occur either at a critical point of $f(x)$ in $(0, \sqrt{6}]$, at $x = \sqrt{6}$, or it will not exist. The critical points of $f(x)$ are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(3x - \frac{1}{2}x^3\right)' &= 0 \\ 3 - \frac{3}{2}x^2 &= 0 \\ x^2 &= 2 \\ x &= \pm\sqrt{2} \end{aligned}$$

However, since $x = -\sqrt{2}$ is outside $(0, \sqrt{6}]$, the only critical point is $x = \sqrt{2}$. Plugging this into $f(x)$ we get:

$$f\left(\sqrt{2}\right) = 3\left(\sqrt{2}\right) - \frac{1}{2}\left(\sqrt{2}\right)^3 = 2\sqrt{2}$$

Evaluating $f(x)$ at $x = \sqrt{6}$ and taking the limit of $f(x)$ as x approaches $x = 0$ we get:

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} \left(3x - \frac{1}{2}x^3\right) = 0 \\ f\left(\sqrt{6}\right) &= 3\left(\sqrt{6}\right) - \frac{1}{2}\left(\sqrt{6}\right)^3 = 0 \end{aligned}$$

both of which are smaller than $2\sqrt{2}$. We conclude that the volume is an absolute maximum at $x = \sqrt{2}$ and that the resulting volume is $\boxed{2\sqrt{2}\ \text{m}^3}$.

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Problem 18 Solution

18. You plan to build a wall enclosing a rectangular garden area of 20,000 square meters. There is a river on the one side, the wall will be built along the other 3 sides. Determine the dimensions that will minimize the length of the wall.

Solution: We begin by letting x be the length of the side opposite the river and y be the lengths of the remaining two sides. The function we seek to minimize is the length of the wall:

$$\textbf{Function}: \qquad \text{Length} = x + 2y \tag{1}$$

The constraint in this problem is that the area of the garden is 20,000 square meters.

$$\textbf{Constraint}: \qquad xy = 20,000 \tag{2}$$

Solving the constraint equation (2) for y we get:

$$y = \frac{20,000}{x} \tag{3}$$

Plugging this into the function (1) and simplifying we get:

$$\begin{aligned} \text{Length} &= x + 2\left(\frac{20,000}{x}\right) \\ f(x) &= x + \frac{40,000}{x} \end{aligned}$$

We want to find the absolute minimum of $f(x)$ on the **interval** $(0, \infty)$. We choose this interval because x must be nonnegative (x represents a length) and non-zero (if x were 0, then the area would be 0 but it must be $20,000$).

The absolute minimum of $f(x)$ will occur either at a critical point of $f(x)$ in $(0, \infty)$ or it will not exist because the interval is open. The critical points of $f(x)$ are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(x + \frac{40,000}{x}\right)' &= 0 \\ 1 - \frac{40,000}{x^2} &= 0 \\ x^2 &= 40,000 \\ x &= \pm 200 \end{aligned}$$

However, since $x = -200$ is outside $(0, \infty)$, the only critical point is $x = 200$. Plugging this into $f(x)$ we get:

$$f(200) = 200 + \frac{40,000}{200} = 400$$

Taking the limits of $f(x)$ as x approaches the endpoints we get:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left(x + \frac{40,000}{x} \right) = 0 + \infty = \infty$$
$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} \left(x + \frac{40,000}{x} \right) = \infty + 0 = \infty$$

both of which are larger than 400. We conclude that the length is an absolute minimum at $\boxed{x = 200}$ and that the resulting length is 400. The last step is to find the corresponding value for y by plugging $x = 200$ into equation (3).

$$y = \frac{20,000}{x} = \frac{20,000}{200} = \boxed{100}$$

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Problem 19 Solution

19. Find the linearization of the function $f(x) = \dfrac{1}{x^2+1}$ at the point $a = 1$.

Solution: The linearization $L(x)$ of the function $f(x)$ at $x = 1$ is defined as:

$$L(x) = f(1) + f'(1)(x-1)$$

The derivative $f'(x)$ is found using the Chain Rule:

$$f'(x) = -\frac{1}{(x^2+1)^2} \cdot \left(x^2+1\right)'$$
$$f'(x) = -\frac{2x}{(x^2+1)^2}$$

For $a = 1$, the values of f' and f are:

$$f'(1) = -\frac{2(1)}{(1^2+1)^2} = -\frac{1}{2}$$
$$f(1) = \frac{1}{1^2+1} = \frac{1}{2}$$

Therefore, the linearization $L(x)$ is:

$$\boxed{L(x) = \frac{1}{2} - \frac{1}{2}(x-1)}$$

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Problem 20 Solution

20. Find antiderivatives of the functions:

(a) $f(x) = 3x^2 - 2x$ (b) $f(x) = e^{3x}$ (c) $f(x) = \dfrac{2}{x^2}$ (d) $f(x) = \cos(2x)$

Solution:

(a) Using the linearity and power rules we have:

$$\begin{aligned}\int f(x)\,dx &= \int \left(3x^2 - 2x\right)\,dx \\ &= 3\int x^2\,dx - 2\int x\,dx \\ &= 3\left(\frac{1}{3}x^3\right) - 2\left(\frac{1}{2}x^2\right) + C \\ &= \boxed{x^3 - x^2 + C}\end{aligned}$$

(b) Using the rule $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$ with $k = 3$, we have:

$$\begin{aligned}\int f(x)\,dx &= \int e^{3x}\,dx \\ &= \boxed{\frac{1}{3}e^{3x} + C}\end{aligned}$$

(c) Using the linearity and power rules we have:

$$\begin{aligned}\int f(x)\,dx &= \int \frac{2}{x^2}\,dx \\ &= 2\int x^{-2}\,dx \\ &= 2\left(\frac{x^{-1}}{-1}\right) + C \\ &= \boxed{-\frac{2}{x} + C}\end{aligned}$$

(d) Using the rule $\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + C$ with $k = 2$ we have:

$$\begin{aligned}\int f(x)\,dx &= \int \cos(2x)\,dx \\ &= \boxed{\frac{1}{2}\sin(2x) + C}\end{aligned}$$