Math 180, Exam 2, Spring 2008
Problem 1 Solution

1. Find the derivatives of the following functions: (do not simplify)

(a) $f(x) = x^2 \cos(x+1)$

(b) $g(x) = \sin(x \ln x)$

(c) $h(x) = \tan^{-1}(2x^2+1)$

(d) $k(x) = (x+x^3)^{1776}$

Solution:

(a) Use the Product Rule and the Chain Rule.

$$\begin{aligned} f'(x) &= [x^2 \cos(x+1)]' \\ &= x^2[\cos(x+1)]' + (x^2)'\cos(x+1) \\ &= x^2\left[-\sin(x+1)\cdot(x+1)'\right] + 2x\cos(x+1) \\ &= \boxed{-x^2\sin(x+1) + 2x\cos(x+1)} \end{aligned}$$

(b) Use the Chain Rule and the Product Rule.

$$\begin{aligned} g'(x) &= [\sin(x \ln x)]' \\ &= \cos(x\ln x)\cdot(x\ln x)' \\ &= \cos(x\ln x)\cdot[x(\ln x)' + (x)'\ln x] \\ &= \cos(x\ln x)\cdot\left(x\cdot\frac{1}{x} + \ln x\right) \\ &= \boxed{(1+\ln x)\cos(x\ln x)} \end{aligned}$$

(c) Use the Chain Rule.

$$\begin{aligned} \left[\tan^{-1}(2x^2+1)\right]' &= \frac{1}{1+(2x^2+1)^2}\cdot(2x^2+1)' \\ &= \boxed{\frac{1}{1+(2x^2+1)^2}\cdot(4x)} \end{aligned}$$

(d) Use the Chain Rule.

$$\begin{aligned} \left[(x+x^3)^{1776}\right]' &= 1776(x+x^3)^{1775}\cdot(x+x^3)' \\ &= \boxed{1776(x+x^3)^{1775}(1+3x^2)} \end{aligned}$$

Math 180, Exam 2, Spring 2008
Problem 2 Solution

2. Let $f(x) = x^3 + 3x^2 - 9x - 11$.

(a) Find the critical points of f.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

(c) Find the local minima and maxima of f. Compute x and $f(x)$ for each local extremum.

(d) Determine the intervals on which f is concave up and the intervals on which f is concave down.

(e) Find the points of inflection of f.

(f) Sketch the graph of f.

Solution:

(a) The critical points of $f(x)$ are the values of x for which either $f'(x)$ does not exist or $f'(x) = 0$. Since $f(x)$ is a polynomial, $f'(x)$ exists for all $x \in \mathbb{R}$ so the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(x^3 + 3x^2 - 9x - 11\right)' &= 0 \\ 3x^2 + 6x - 9 &= 0 \\ 3(x^2 + 2x - 3) &= 0 \\ 3(x+3)(x-1) &= 0 \\ x = -3,\ x &= 1 \end{aligned}$$

Thus, $\boxed{x = -3}$ and $\boxed{x = 1}$ are the critical points of f.

(b) The domain of f is $(-\infty, \infty)$. We now split the domain into the three intervals $(-\infty, -3)$, $(-3, 1)$, and $(1, \infty)$. We then evaluate $f'(x)$ at a test point in each interval to determine the intervals of monotonicity.

Interval	**Test Point**, c	$f'(c)$	**Sign of** $f'(c)$
$(-\infty, -3)$	-4	$f'(-4) = 15$	$+$
$(-3, 1)$	0	$f'(0) = -9$	$-$
$(1, \infty)$	2	$f'(2) = 15$	$+$

Using the table, we conclude that f is increasing on $(-\infty, -3) \cup (1, \infty)$ because $f'(x) > 0$ for all $x \in (-\infty, -3) \cup (1, \infty)$ and f is decreasing on $(-3, 1)$ because $f'(x) < 0$ for all $x \in (-3, 1)$.

(c) Since f' changes sign from $+$ to $-$ at $x = -3$ the First Derivative Test implies that $\boxed{f(-3) = 16}$ is a local maximum and since f' changes sign from $-$ to $+$ at $x = 1$ the First Derivative Test implies that $\boxed{f(1) = -16}$ is a local minimum.

(d) To determine the intervals of concavity we start by finding solutions to the equation $f''(x) = 0$ and where $f''(x)$ does not exist. However, since $f(x)$ is a polynomial we know that $f''(x)$ will exist for all $x \in \mathbb{R}$. The solutions to $f''(x) = 0$ are:

$$\begin{aligned} f''(x) &= 0 \\ \left(3x^2 + 6x - 9\right)' &= 0 \\ 6x + 6 &= 0 \\ x &= -1 \end{aligned}$$

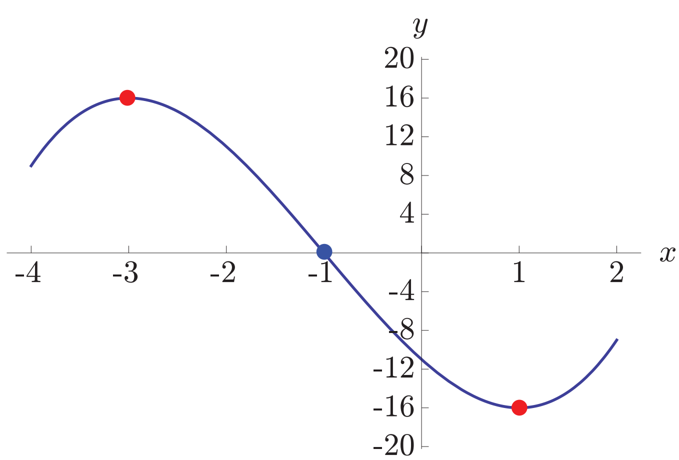
We now split the domain into the two intervals $(-\infty, -1)$ and $(-1, \infty)$. We then evaluate $f''(x)$ at a test point in each interval to determine the intervals of concavity.

Interval	**Test Point**, c	$f'(c)$	**Sign of** $f'(c)$
$(-\infty, -1)$	-2	$f''(-2) = -6$	$-$
$(-1, \infty)$	0	$f''(0) = 6$	$+$

Using the table, we conclude that f is concave down on $(-\infty, -1)$ because $f''(x) < 0$ for all $x \in (-\infty, -1)$ and f is concave up on $(-1, \infty)$ because $f''(x) > 0$ for all $x \in (-1, \infty)$.

(e) The inflection points of $f(x)$ are the points where $f''(x)$ changes sign. We can see in the above table that $f''(x)$ changes sign at $x = -1$. Therefore, $\boxed{x = -1}$ is an inflection point.

(f)



Math 180, Exam 2, Spring 2008
Problem 3 Solution

3. Find an equation for the tangent line to the curve defined by the equation $x^2y^2+xy+y=1$ at the point $(2,-1)$.

Solution: To find y', the slope of the tangent line, we use implicit differentiation.

$$\begin{aligned} x^2y^2+xy+y &= 1 \\ (x^2y^2)'+(xy)'+(y)' &= (1)' \\ \big[(x^2)(y^2)'+(y^2)(x^2)'\big]+[(x)(y)'+(y)(x)']+y' &= 0 \\ \big[(x^2)(2yy')+(y^2)(2x)\big]+[(x)(y')+(y)(1)]+y' &= 0 \\ 2x^2yy'+2xy^2+xy'+y+y' &= 0 \\ 2x^2yy'+xy'+y' &= -2xy^2-y \\ y'(2x^2y+x+1) &= -2xy^2-y \\ y' &= \frac{-2xy^2-y}{2x^2y+x+1} \end{aligned}$$

At the point $(2,-1)$, the value of y' is:

$$y'(2,-1)=\frac{-2(2)(-1)^2-(-1)}{2(2)^2(-1)+2+1}=\frac{3}{5}$$

An equation for the tangent line is then:

$$\boxed{y+1=\frac{3}{5}(x-2)}$$

Math 180, Exam 2, Spring 2008
Problem 4 Solution

4. Find the minimum and the maximum values on the interval $[0, 5]$ for the function:

$$f(x) = x^3 - 9x^2 + 24x + 1$$

Solution: The minimum and maximum values of $f(x)$ will occur at a critical point in the interval $[0, 5]$ or at one of the endpoints of the interval. The critical points are the values of x for which either $f'(x) = 0$ or $f'(x)$ does not exist. Since $f(x)$ is a polynomial, $f'(x)$ exists for all $x \in \mathbb{R}$. Therefore, the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(x^3 - 9x^2 + 24x + 1\right)' &= 0 \\ 3x^2 - 18x + 24 &= 0 \\ 3(x^2 - 6x + 8) &= 0 \\ 3(x-2)(x-4) &= 0 \\ x = 2,\ x &= 4 \end{aligned}$$

The critical points $x = 2$ and $x = 4$ both lie in $[0, 5]$. Therefore, we check the value of $f(x)$ at $x = 0, 2, 4$, and 5.

$$\begin{aligned} f(0) &= 0^3 - 9(0)^2 + 24(0) + 1 = 1 \\ f(2) &= 2^3 - 9(2)^2 + 24(2) + 1 = 21 \\ f(4) &= 4^3 - 9(4)^2 + 24(4) + 1 = 17 \\ f(5) &= 5^3 - 9(5)^2 + 24(5) + 1 = 21 \end{aligned}$$

The minimum value of $f(x)$ on $[0, 5]$ is $\boxed{1}$ because it is the smallest of the above values of f. The maximum is $\boxed{21}$ because it is the largest.

Math 180, Exam 2, Spring 2008
Problem 5 Solution

5. You are designing a closed cardboard box (with a top, a bottom, and four sides). The length of the box must be twice its width and the volume of the box must be 72 cubic inches.

(a) Express the surface area of the box as a function of its width.

(b) Determine the dimensions of the box that will use the least amount of cardboard.

Solution:

(a) We begin by letting x be the width of the box, y be the height, and z be the length. The surface area then has the equation:

$$\text{Surface Area} = 2xy + 2yz + 2xz \tag{1}$$

We are asked to write the surface area as a function of width, x. Therefore, we must find equations that relate x to y and z so that we can eliminate the latter two variables from the surface area equation.

One of the constraints in the problem is that the length must be twice its width. This gives us the equation

$$z = 2x \tag{2}$$

The other constraint in the problem is that the volume must be 72 cubic inches. This gives us the equation

$$\text{Volume} = xyz = 72 \tag{3}$$

Plugging equation (2) into equation (3) and solving for y we get

$$\begin{aligned} xyz &= 72 \\ xy(2x) &= 72 \\ y &= \frac{36}{x^2} \qquad (4) \end{aligned}$$

Finally, we write the surface area as a function of x by plugging equations (2) and (4) into equation (1).

$$\begin{aligned} \text{Surface Area} &= 2xy + 2yz + 2xz \\ \text{Surface Area} &= 2x\left(\frac{36}{x^2}\right) + 2\left(\frac{36}{x^2}\right)(2x) + 2x(2x) \\ \text{Surface Area} &= \frac{216}{x} + 4x^2 \qquad (5) \\ &\boxed{f(x) = \frac{216}{x} + 4x^2} \qquad (6) \end{aligned}$$

(b) The surface area represents the amount of cardboard necessary to construct the box. Therefore, we seek the value of x that minimizes $f(x)$. The interval in the problem is $(0, \infty)$ because x must be nonnegative (it is a length quantity) and nonzero (the volume must be 72).

The absolute minimum of $f(x)$ will occur either at a critical point of $f(x)$ in $(0, \infty)$ or it will not exist because the interval is open. The critical points of $f(x)$ are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(\frac{216}{x} + 4x^2\right)' &= 0 \\ -\frac{216}{x^2} + 8x &= 0 \\ 8x^3 - 216 &= 0 \\ x^3 &= \frac{216}{8} \\ \boxed{x = 3} \end{aligned}$$

Plugging this into $f(x)$ we get:

$$f(3) = \frac{216}{3} + 4(3)^2 = 108$$

Taking the limits of $f(x)$ as x approaches the endpoints we get:

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} \left(\frac{216}{x} + 4x^2\right) = \infty + 0 = \infty \\ \lim_{x \to \infty} f(x) &= \lim_{x \to \infty} \left(\frac{216}{x} + 4x^2\right) = 0 + \infty = \infty \end{aligned}$$

both of which are larger than 108. We conclude that the cost is an absolute minimum at $x = 3$ and that the resulting surface area is 108. The last step is to find the corresponding values for y and z by plugging $x = 3$ into equations (4) and (2).

$$\begin{aligned} z &= 2x = 2(3) = \boxed{6} \\ y &= \frac{36}{x^2} = \frac{36}{3^2} = \boxed{4} \end{aligned}$$

Math 180, Exam 2, Spring 2008
Problem 6 Solution

6. Use L'Hôpital's Rule to compute the limit:

$$\lim_{x\to 0} \frac{\sin(3x^2)}{1-e^{x^2}}$$

Solution: Upon substituting $x = 0$ into the function $\frac{\sin(3x^2)}{1-e^{x^2}}$ we find that

$$\frac{\sin(3(0)^2)}{1-e^{0^2}} = \frac{0}{0}$$

which is indeterminate. We resolve the indeterminacy using L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(3x^2)}{1-e^{x^2}} &\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{(\sin(3x^2))'}{(1-e^{x^2})'} \\
&= \lim_{x\to 0} \frac{6x\cos(3x^2)}{-2xe^{x^2}} \\
&= \lim_{x\to 0} \frac{3\cos(3x^2)}{-e^{x^2}} \\
&= \frac{3\cos(3(0)^2)}{-e^{0^2}} \\
&= \boxed{-3}
\end{aligned}$$