Math 180, Exam 2, Spring 2009
Problem 1 Solution

1. Compute the derivative of the following functions:

(a) $f(x) = x\ln(x^2+1)$

(b) $g(x) = \cos(xe^x)$

(c) $h(x) = \tan^{-1}(2x+1)$

Solution:

(a) Use the Product and Chain Rules.

$$\begin{aligned} f'(x) &= x[\ln(x^2+1)]' + (x)'\ln(x^2+1) \\ &= x \cdot \frac{1}{x^2+1} \cdot (x^2+1)' + 1 \cdot \ln(x^2+1) \\ &= x \cdot \frac{1}{x^2+1} \cdot 2x + \ln(x^2+1) \\ &= \boxed{\frac{2x^2}{x^2+1} + \ln(x^2+1)} \end{aligned}$$

(b) Use the Chain and Product Rules.

$$\begin{aligned} g'(x) &= -\sin(xe^x) \cdot (xe^x)' \\ &= -\sin(xe^x) \cdot [x(e^x)' + (x)'e^x] \\ &= \boxed{-\sin(xe^x) \cdot (xe^x + e^x)} \end{aligned}$$

(c) Use the Chain Rule.

$$\begin{aligned} h'(x) &= \frac{1}{1+(2x+1)^2} \cdot (2x+1)' \\ &= \boxed{\frac{1}{1+(2x+1)^2} \cdot 2} \end{aligned}$$

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Problem 2 Solution

2. Use L'Hôpital's Rule to compute the limit:

$$\lim_{x\to\infty} \frac{\ln(x^2+1)}{x+\sqrt{x}}$$

Solution: As $x \to \infty$, the function $\frac{\ln(x^2+1)}{x+\sqrt{x}} \to \frac{\infty}{\infty}$ which is indeterminate. We resolve the indeterminacy using L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln(x^2+1)}{x+\sqrt{x}} &\overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{(\ln(x^2+1))'}{(x+\sqrt{x})'} \\
&= \lim_{x\to\infty} \frac{\frac{1}{x^2+1}\cdot 2x}{1+\frac{1}{2\sqrt{x}}} \\
&= \lim_{x\to\infty} \frac{\frac{1}{x^2+1}\cdot 2x}{1+\frac{1}{2\sqrt{x}}} \cdot \frac{2\sqrt{x}(x^2+1)}{2\sqrt{x}(x^2+1)} \\
&= \lim_{x\to\infty} \frac{4x\sqrt{x}}{(2\sqrt{x}+1)(x^2+1)} \\
&= \lim_{x\to\infty} \frac{4x^{3/2}}{2x^{5/2}+x^2+2x^{1/2}+1} \\
&= \lim_{x\to\infty} \frac{4x^{3/2}}{2x^{5/2}} \\
&= \lim_{x\to\infty} \frac{2}{x} \\
&= \boxed{0}
\end{aligned}$$

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Problem 3 Solution

3. Consider the curve defined implicitly by $x^3 + y^3 - 9xy = 0$.

(a) Show that the point $(2, 4)$ lies on the curve.

(b) Find an equation for the line tangent to the curve at $(2, 4)$.

Solution:

(a) To show that the point $(2, 4)$ lies on the curve, plug these values for x and y into the equation.

$$\begin{aligned} x^3 + y^3 - 9xy &= 0 \\ 2^3 + 4^3 - 9(2)(4) &= 0 \\ 8 + 64 - 72 &= 0 \\ 0 &= 0 \end{aligned}$$

Since we get $0 = 0$, the point lies on the curve.

(b) To find the slope of the tangent line, use implicit differentiation to find y'.

$$\begin{aligned} x^3 + y^3 - 9xy &= 0 \\ (x^3)' + (y^3)' - (9xy)' &= (0)' \\ 3x^2 + 3y^2y' - [(9x)(y)' + (y)(9x)'] &= 0 \\ 3x^2 + 3y^2y' - [9xy' + 9y] &= 0 \\ 3y^2y' - 9xy' &= -3x^2 + 9y \\ y'(3y^2 - 9x) &= -3x^2 + 9y \\ y' &= \frac{-3x^2 + 9y}{3y^2 - 9x} \\ y' &= \frac{-x^2 + 3y}{y^2 - 3x} \end{aligned}$$

At the point $(2, 4)$, the value of y' is:

$$y'(2, 4) = \frac{-2^2 + 3(4)}{4^2 - 3(2)} = \frac{4}{5}$$

An equation for the tangent line is then:

$$\boxed{y - 4 = \frac{4}{5}(x - 2)}$$

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Problem 4 Solution

4. Let $f(x) = x^3 + 3x^2 - 9x + 5$.

(a) Find the critical points of f and classify each as a local minimum, a local maximum, or neither.

(b) On what interval(s) is f concave down?

(c) Find the absolute minimum of f over the interval $[-2, 1]$.

Solution:

(a) The critical points of $f(x)$ are the values of x for which either $f'(x) = 0$ or $f'(x)$ does not exist. Since $f(x)$ is a polynomial, $f'(x)$ exists for all $x \in \mathbb{R}$. Therefore, the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(x^3 + 3x^2 - 9x + 5\right)' &= 0 \\ 3x^2 + 6x - 9 &= 0 \\ 3(x^2 + 2x - 3) &= 0 \\ 3(x+3)(x-1) &= 0 \\ x = -3,\ x &= 1 \end{aligned}$$

We use the Second Derivative Test to classify the critical points $\boxed{x = -3}$ and $\boxed{x = 1}$. The second derivative is $f''(x) = 6x + 6$. At the critical points, we have:

$$\begin{aligned} f''(-3) &= 6(-3) + 6 = -12 \\ f''(1) &= 6(1) + 6 = 12 \end{aligned}$$

Since $f''(-3) < 0$ the Second Derivative Test implies that $f(-3) = 32$ is a local maximum. Since $f''(1) > 0$ the Second Derivative Test implies that $f(1) = 0$ is a local minimum.

(b) A function $f(x)$ is concave down on (a, b) when $f''(x) < 0$ for all $x \in (a, b)$. To find the interval(s) where f is concave down, we must first determine the value(s) of x for which $f''(x) = 0$.

$$\begin{aligned} f''(x) &= 0 \\ \left(3x^2 + 6x - 9\right)' &= 0 \\ 6x + 6 &= 0 \\ x &= -1 \end{aligned}$$

Since the domain of f is $(-\infty, \infty)$, we divide the domain into the two intervals $(-\infty, -1)$ and $(-1, \infty)$. We now evaluate f'' at test points in each interval to determine where $f''(x)$ is positive and negative.

Interval	**Test Number, c**	$f''(c)$	**Sign of $f''(c)$**
$(-\infty, -1)$	-2	-6	$-$
$(-1, \infty)$	0	6	$+$

Since $f''(-2) = -6 < 0$, we know that f is concave down on the interval $\boxed{(-\infty, -1)}$.

(c) The absolute minimum of f will occur either at a critical point in $[-2, 1]$ or at one of the endpoints. From part (a), we found that the critical points of f are $x = -3$ and $x = 1$. The point $x = -3$ is outside the interval and $x = 1$ is an endpoint. Therefore, we only evaluate f at $x = -2$ and $x = 1$.

$$f(-2) = (-2)^3 + 3(-2)^2 - 9(-2) + 5 = 27$$
$$f(1) = 1^3 + 3(1)^2 - 9(1) + 5 = 0$$

The absolute minimum of f on $[-2, 1]$ is $\boxed{0}$ because it is the smallest of the values of f above.

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Problem 5 Solution

5. The sum of two nonnegative numbers is 36. Find the numbers if the sum of their square roots is to be as large as possible.

Solution: We begin by letting x and y be the numbers in question. The function we seek to minimize is:

$$\textbf{Function}: \quad \text{sum} = \sqrt{x} + \sqrt{y} \tag{1}$$

The constraint in this problem is that the sum of x and y must be 36.

$$\textbf{Constraint}: \quad x + y = 36 \tag{2}$$

Solving the constraint equation (2) for y we get:

$$y = 36 - x \tag{3}$$

Plugging this into the function (1) we get:

$$\begin{aligned} \text{sum} &= \sqrt{x} + \sqrt{36 - x} \\ f(x) &= \sqrt{x} + \sqrt{36 - x} \end{aligned}$$

We want to find the absolute maximum of $f(x)$ on the **interval** $[0, 36]$. We choose this interval because x must be nonnegative $(0 \le x)$ and the sum of x and y must be 36 $(x \le 36)$.

The absolute minimum of $f(x)$ will occur either at a critical point of $f(x)$ in $[0, 36]$ or at one of the endpoints. The critical points of $f(x)$ are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{36 - x}} &= 0 \\ \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{36 - x}} &= 0 \\ \frac{1}{\sqrt{x}} &= \frac{1}{\sqrt{36 - x}} \\ \sqrt{36 - x} &= \sqrt{x} \\ 36 - x &= x \\ 2x &= 36 \\ x &= 18 \end{aligned}$$

Plugging this into $f(x)$ we get:

$$f(18) = \sqrt{18} + \sqrt{36 - 18} = 6\sqrt{2}$$

Evaluating $f(x)$ at the endpoints $x = 0$ and $x = 36$ we get:

$$f(0) = \sqrt{0} + \sqrt{36 - 0} = 6$$
$$f(36) = \sqrt{36} + \sqrt{36 - 36} = 6$$

both of which are smaller than $6\sqrt{2}$. We conclude that the sum is an absolute maximum at $\boxed{x = 18}$ and that the resulting cost is $6\sqrt{2}$. The last step is to find the corresponding value for y by plugging $x = 18$ into equation (3).

$$\boxed{y = 36 - 18 = 18}$$