Math 180, Exam 2, Spring 2010
Problem 1 Solution

1. Compute the indefinite integrals:

(a) $\displaystyle\int \left(x^3 - 4x^2 + 3x + 5\right)\, dx$

(b) $\displaystyle\int \sqrt{x}\left(x^2 - 1\right)\, dx$

Solution:

(a) Using the linearity and power rules we have:

$$\begin{aligned}\int \left(x^3 - 4x^2 + 3x + 5\right)\, dx &= \int x^3\, dx - 4\int x^2\, dx + 3\int x\, dx + 5\int dx \\ &= \frac{1}{4}x^3 - 4\left(\frac{1}{3}x^3\right) + 3\left(\frac{1}{2}x^2\right) + 5(x) + C \\ &= \boxed{\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2 + 5x + C}\end{aligned}$$

(b) Using some algebra and the linearity and power rules we have:

$$\begin{aligned}\int \sqrt{x}\left(x^2 - 1\right)\, dx &= \int \left(x^{5/2} - x^{1/2}\right)\, dx \\ &= \int x^{5/2}\, dx - \int x^{1/2}\, dx \\ &= \boxed{\frac{2}{7}x^{7/2} - \frac{2}{3}x^{3/2} + C}\end{aligned}$$

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Problem 2 Solution

2. Use L'Hôpital's Rule to compute $\lim_{x \to 0} \dfrac{e^{7x} - 1}{e^{3x} - 1}$.

Solution: Upon substituting $x = 0$ into the function $\frac{e^{7x}-1}{e^{3x}-1}$ we get

$$\frac{e^{7(0)} - 1}{e^{3(0)} - 1} = \frac{0}{0}$$

which is indeterminate. We resolve the indeterminacy using L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x \to 0} \frac{e^{7x} - 1}{e^{3x} - 1} &\overset{\text{L'H}}{=} \lim_{x \to 0} \frac{(e^{7x} - 1)'}{(e^{3x} - 1)'} \\
&= \lim_{x \to 0} \frac{7e^{7x}}{3e^{3x}} \\
&= \frac{7e^{7(0)}}{3e^{3(0)}} \\
&= \boxed{\frac{7}{3}}
\end{aligned}$$

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Problem 3 Solution

3. Let $f(x) = x^3 - 2x^2 + x$.

(a) Find the critical point(s) of f and classify each as a local maximum, local minimum, or neither. Determine the intervals of monotonicity of f.

(b) Find the inflection point(s) of f. Determine the intervals where f is concave up and concave down.

(c) Sketch the graph $y = f(x)$, labeling the critical points and inflection points.

Solution:

(a) The critical points of $f(x)$ are the values of x for which either $f'(x)$ does not exist or $f'(x) = 0$. Since $f(x)$ is a polynomial, $f'(x)$ exists for all $x \in \mathbb{R}$ so the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(x^3 - 2x^2 + x\right)' &= 0 \\ 3x^2 - 4x + 1 &= 0 \\ (3x-1)(x-1) &= 0 \\ x = \frac{1}{3},\ x &= 1 \end{aligned}$$

Thus, $\boxed{x = \frac{1}{3}}$ and $\boxed{x = 1}$ are the critical points of f.

We will use the First Derivative Test to classify the critical points. The domain of f is $(-\infty, \infty)$. We now split the domain into the three intervals $(-\infty, \frac{1}{3})$, $(\frac{1}{3}, 1)$, and $(1, \infty)$. We then evaluate $f'(x)$ at a test point in each interval.

Interval	**Test Point**, c	$f'(c)$	**Sign of** $f'(c)$
$(-\infty, \frac{1}{3})$	0	$f'(0) = 1$	$+$
$(\frac{1}{3}, 1)$	$\frac{2}{3}$	$f'(\frac{2}{3}) = -\frac{1}{3}$	$-$
$(1, \infty)$	2	$f'(2) = 5$	$+$

Since the sign of $f'(x)$ changes from $+$ to $-$ at $x = \frac{1}{3}$, the First Derivative Test implies that $f(\frac{1}{3}) = \frac{4}{27}$ is a local maximum. Since the sign of $f'(x)$ changes from $-$ to $+$ at $x = 1$, the First Derivative Test implies that $f(1) = 0$ is a local minimum. Furthermore, from the table we conclude that f is increasing on $(-\infty, \frac{1}{3}) \cup (1, \infty)$ because $f'(x) > 0$ for all $x \in (-\infty, \frac{1}{3}) \cup (1, \infty)$ and f is decreasing on $(\frac{1}{3}, 1)$ because $f'(x) < 0$ for all $x \in (\frac{1}{3}, 1)$.

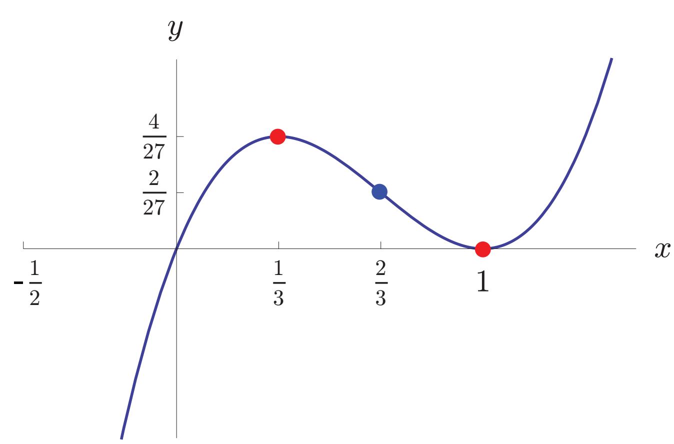
(b) The inflection points of $f(x)$ are the points where $f''(x)$ changes sign. To determine these points we start by finding solutions to the equation $f''(x) = 0$.

$$\begin{aligned} f''(x) &= 0 \\ \left(3x^2 + 4x + 1\right)' &= 0 \\ 6x - 4 &= 0 \\ x &= \frac{2}{3} \end{aligned}$$

We now split the domain of f into the two intervals $(-\infty, \frac{2}{3})$ and $(\frac{2}{3}, \infty)$. We then evaluate $f''(x)$ at a test point in each interval to determine the intervals of concavity.

Interval	**Test Point**, c	$f'(c)$	**Sign of** $f'(c)$
$(-\infty, \frac{2}{3})$	0	$f''(0) = -4$	$-$
$(\frac{2}{3}, \infty)$	1	$f''(1) = 2$	$+$

Since there is a sign change in $f''(x)$ at $\boxed{x = \frac{2}{3}}$, the point $x = \frac{2}{3}$ is an inflection point. Furthermore, from the table we conclude that f is concave up on $(\frac{2}{3}, \infty)$ because $f''(x) > 0$ for all $x \in (\frac{2}{3}, \infty)$ and f is concave down on $(-\infty, \frac{2}{3})$ because $f''(x) < 0$ for all $x \in (-\infty, \frac{2}{3})$.



(c)

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Problem 4 Solution

4. Find the absolute minimum and the absolute maximum of $f(x) = x^3/3 - x^2/2 + 2$ on the interval $[-1, 2]$.

Solution: The minimum and maximum values of $f(x)$ will occur at a critical point in the interval $[-1, 2]$ or at one of the endpoints. The critical points are the values of x for which either $f'(x) = 0$ or $f'(x)$ does not exist. Since $f(x)$ is a polynomial, $f'(x)$ exists for all $x \in \mathbb{R}$. Therefore, the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(x^3/3 - x^2/2 + 2\right)' &= 0 \\ x^2 - x &= 0 \\ x(x-1) &= 0 \\ x = 0,\ x &= 1 \end{aligned}$$

Both critical points $x = 0$ and $x = 1$ lie in $[-1, 2]$. Therefore, we check the value of $f(x)$ at $x = -1,\ 0,\ 1,$ and 2.

$$\begin{aligned} f(-1) &= (-1)^3/3 - (-1)^2/2 + 2 = \frac{7}{6} \\ f(0) &= 0^3/3 - 0^2/2 + 2 = 2 \\ f(1) &= 1^3/3 - 1^2/2 + 2 = \frac{11}{6} \\ f(2) &= 2^3/3 - 2^2/2 + 2 = \frac{8}{3} \end{aligned}$$

The minimum value of $f(x)$ on $[-1, 2]$ is $\boxed{\frac{7}{6}}$ because it is the smallest of the above values of f. The maximum is $\boxed{\frac{8}{3}}$ because it is the largest.

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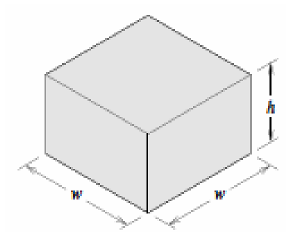
Problem 5 Solution

5. Design a rectangular box with square base (as in the diagram below) and a total surface area of 6 square feet that encloses the maximum possible volume. Determine both the dimensions of the box and the volume enclosed.



Solution: We begin by letting w be the length of one side of the base and h be the height of the box. The function we seek to minimize is the volume of the box.

$$\textbf{Function}: \quad \text{Volume} = w^2h \tag{1}$$

The constraint in the problem is that the total surface area is 6. This gives us the equation

$$\textbf{Constraint}: \quad 2w^2 + 4wh = 6 \tag{2}$$

Solving this equation for h we get

$$\begin{aligned} 2w^2 + 4wh &= 6 \\ w^2 + 2wh &= 3 \\ h &= \frac{3 - w^2}{2w} \end{aligned} \tag{3}$$

We then plug this into the volume equation (1) to write the volume in terms of w only.

$$\begin{aligned} \text{Volume} &= w^2h \\ \text{Volume} &= w^2\left(\frac{3 - w^2}{2w}\right) \\ f(w) &= \frac{3}{2}w - \frac{1}{2}w^3 \end{aligned} \tag{4}$$

We want to find the absolute maximum of $f(w)$ on the interval $\left(0, \sqrt{3}\right]$. We know that $w > 0$ because w must be positive and nonzero (otherwise, the surface area would be 0 and it must be 6). It is possible that $h = 0$ in which case the surface area constraint would give us $2w^2 + 4w(0) = 6 \quad \Rightarrow \quad w^2 = 3 \quad \Rightarrow \quad w = \sqrt{3}$.

The absolute maximum of $f(w)$ will occur either at a critical point of $f(w)$ in $\left(0, \sqrt{3}\right]$, at $x = \sqrt{w}$, or it will not exist. The critical points of $f(w)$ are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(w) &= 0 \\ \frac{3}{2} - \frac{3}{2}w^2 &= 0 \\ w^2 &= 1 \\ w &= \pm 1 \end{aligned}$$

However, since $w = -1$ is outside $\left(0, \sqrt{3}\right]$, the only critical point is $w = 1$. Plugging this into $f(w)$ we get:

$$f(1) = \frac{3}{2}(1) - \frac{1}{2}(1)^2 = 1$$

Evaluating $f(w)$ at $w = \sqrt{3}$ and taking the limit of $f(w)$ as w approaches $w = 0$ we get:

$$\begin{aligned} \lim_{w \to 0^+} f(w) &= \lim_{w \to 0^+} \left(\frac{3}{2}w - \frac{1}{2}w^3 \right) = 0 \\ f\left(\sqrt{3}\right) &= \frac{3}{2}\left(\sqrt{3}\right) - \frac{1}{2}\left(\sqrt{3}\right)^3 = 0 \end{aligned}$$

both of which are smaller than 1. We conclude that the volume is an absolute maximum at $\boxed{w = 1}$ and that the resulting volume is 1 ft^3. The height of the box when $w = 1$ is found using equation (3).

$$\boxed{h = \frac{3 - 1^2}{2(1)} = 1}$$

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Problem 6 Solution

6. Compute the area of the region defined by $2 \le x \le 5$, $0 \le y \le x^2$.

Solution: The area of the region is given by the formula:

$$\text{Area} = \int_2^5 x^2\,dx$$

Using the Fundamental Theorem of Calculus, Part I to evaluate the integral we get:

$$\begin{aligned}\text{Area} &= \int_2^5 x^2\,dx \\ &= \left[\frac{1}{3}x^3\right]_2^5 \\ &= \frac{1}{3}5^3 - \frac{1}{3}2^3 \\ &= \boxed{39}\end{aligned}$$