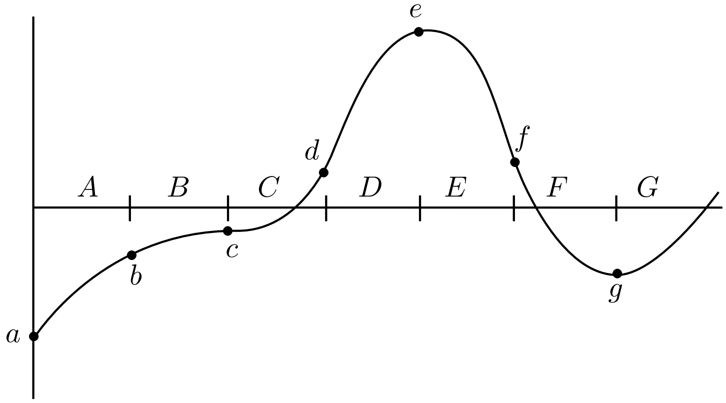
Math 180, Exam 2, Spring 2011
Problem 1 Solution

1. The graph of a function $f(x)$ is shown below:



(a) Fill in the table below with the signs of the first and second derivatives of f on each of the intervals $A, \ldots, G$.

	A	B	C	D	E	F	G
sign of f'							
sign of f''							

(b) Which of the points $a, \ldots, g$ are critical points? For each critical point, say whether it is a local maximum, a local minimum or neither.

(c) Which of the points $a, \ldots, g$ are inflection points?

Solution:

(a)

	A	B	C	D	E	F	G
sign of f'	$+$	$+$	$+$	$+$	$-$	$-$	$+$
sign of f''	$-$	$-$	$+$	$-$	$-$	$+$	$+$

(b) c, e, and g are critical points because $f'(x) = 0$ at these points. c is neither a local minimum nor a local maximum. e is a local maximum. g is a local minimum.

(c) c, d, and f are inflection points because $f''(x)$ changes sign at these points.

Math 180, Exam 2, Spring 2011
Problem 2 Solution

2. Sketch the graph of the function $f(x) = x^{-2} - x^2$ following the steps below.

(a) Determine the domain of f and find all asymptotes.

(b) Find the intervals where the graph of f is increasing, decreasing, concave up and concave down.

(c) Sketch the graph of f, clearly showing any local extrema, inflection points, x-intercepts, y-intercepts and asymptotes.

Solution:

(a) The domain of f is all real numbers except $x = 0$. In fact, $x = 0$ is a vertical asymptote. Furthermore, since

$$\lim_{x\to\pm\infty} \left(x^{-2} - x^2\right) = -\infty$$

we know that f does not have a horizontal asymptote.

(b) To determine the intervals of monotonicity, we first find the critical points of f. These are points where either $f'(x) = 0$ or $f'(x)$ does not exist. The first derivative is:

$$f'(x) = -2x^{-3} - 2x = -2x - \frac{2}{x^3}$$

$f'(x)$ will not exist only when $x = 0$. However, $x = 0$ is not in the domain of f. Therefore, the only critical points we have are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ -2x - \frac{2}{x^3} &= 0 \\ -2\left(x + \frac{1}{x^3}\right) &= 0 \\ -2\left(\frac{x^4+1}{x^3}\right) &= 0 \\ x^4 + 1 &= 0 \end{aligned}$$

This equation has no solutions. Thus, f has no critical points. To determine the intervals of monotonicity we take the domain of f and evaluate $f'(x)$ at test points to determine the sign of $f'(x)$.

Interval	**Test Point**, c	$f'(c)$	**Sign of** $f'(c)$
$(-\infty, 0)$	-1	$f'(-1) = 4$	$+$
$(0, \infty)$	1	$f'(1) = -4$	$-$

From the table we determine that f is decreasing on $(0, \infty)$ because $f'(x) < 0$ for all $x \in (0, \infty)$ and f is increasing on $(-\infty, 0)$ because $f'(x) > 0$ for all $x \in (-\infty, 0)$.

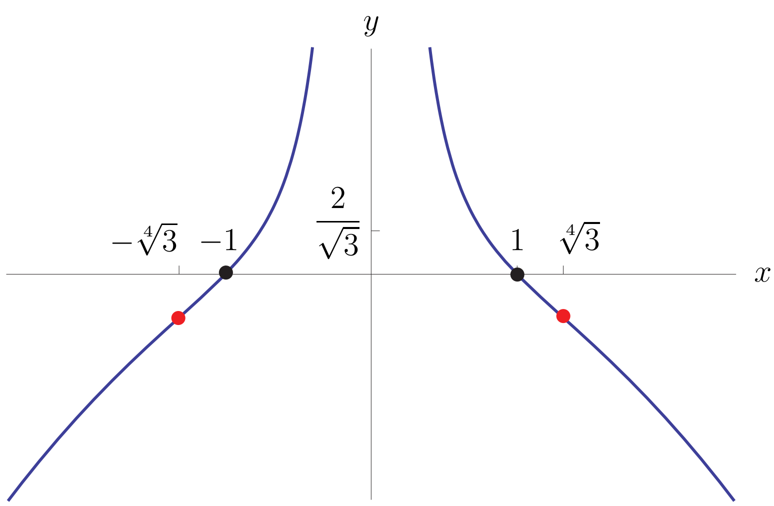
To determine the intervals of concavity, we first find the values of x for which $f''(x) = 0$.

$$
\begin{aligned}
f''(x) &= 0 \\
\left(-2x - 2x^{-3}\right)' &= 0 \\
-2 + 6x^{-4} &= 0 \\
-2 + \frac{6}{x^4} &= 0 \\
-2x^4 + 6 &= 0 \\
x^4 &= 3 \\
x &= \pm\sqrt[4]{3}
\end{aligned}
$$

To determine the intervals of monotonicity we split the domain of f into the intervals $(-\infty, -\sqrt[4]{3})$, $(-\sqrt[4]{3}, 0)$, $(0, \sqrt[4]{3})$, and $(\sqrt[4]{3}, \infty)$ and evaluate $f''(x)$ at test points in each interval to determine the sign of $f''(x)$.

Interval	**Test Point**, c	$f''(c)$	**Sign of** $f''(c)$
$(-\infty, -\sqrt[4]{3})$	-2	$f''(-2) = -\frac{13}{8}$	$-$
$(-\sqrt[4]{3}, 0)$	-1	$f''(-1) = 4$	$+$
$(0, \sqrt[4]{3})$	1	$f''(1) = 4$	$+$
$(\sqrt[4]{3}, \infty)$	2	$f''(2) = -\frac{13}{8}$	$-$

From the table we determine that f is concave down on $(-\infty, -\sqrt[4]{3}) \cup (\sqrt[4]{3}, \infty)$ because $f''(x) < 0$ for all $x \in (-\infty, -\sqrt[4]{3}) \cup (\sqrt[4]{3}, \infty)$ and f is concave up on $(-\sqrt[4]{3}, 0) \cup (0, \sqrt[4]{3})$ because $f''(x) > 0$ for all $x \in (-\sqrt[4]{3}, 0) \cup (0, \sqrt[4]{3})$.



Math 180, Exam 2, Spring 2011
Problem 3 Solution

3. Find the area of the largest rectangle that can be inscribed in a semicircle of radius 3.

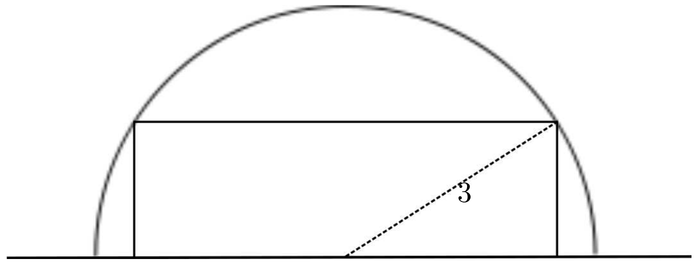


Figure 1:

Solution: Let x be half of the width of the rectangle and let y be the height. We seek to maximize the area so the function is:

$$\text{Area} = 2xy$$

The variables x and y can be related using the Pythagorean Theorem.

$$x^2 + y^2 = 3^2$$

Solving the above equation for y we get:

$$y = \sqrt{9 - x^2}$$

Plugging this into the area formula we get:

$$\begin{aligned} \text{Area} &= 2xy \\ f(x) &= 2x\sqrt{9 - x^2} \end{aligned}$$

We must now find the absolute maximum of $f(x)$ on the domain $[0, 3]$. We start by finding the critical points, which will be the values of x for which $f'(x) = 0$.

$$\begin{aligned}
f'(x) &= 0 \\
\left(2x\sqrt{9-x^2}\right)' &= 0 \\
2x\left(\sqrt{9-x^2}\right)' + (2x)'\sqrt{9-x^2} &= 0 \\
2x\left(\frac{1}{2}(9-x^2)^{-1/2}\cdot(-2x)\right) + 2\sqrt{9-x^2} &= 0 \\
-\frac{2x^2}{\sqrt{9-x^2}} + 2\sqrt{9-x^2} &= 0 \\
\frac{-2x^2 + 2(9-x^2)}{\sqrt{9-x^2}} &= 0 \\
-2x^2 + 2(9-x^2) &= 0 \\
-x^2 + 9 - x^2 &= 0 \\
x^2 &= \frac{9}{2} \\
x &= \frac{3}{\sqrt{2}}
\end{aligned}$$

Evaluating $f(x)$ at $x = 0, \frac{3}{\sqrt{2}}$, and 3 we get:

$$\begin{aligned}
f(0) &= 2(0)\sqrt{9-0^2} = 0 \\
f\left(\frac{3}{\sqrt{2}}\right) &= 2\left(\frac{3}{\sqrt{2}}\right)\sqrt{9-\left(\frac{3}{\sqrt{2}}\right)^2} = 9 \\
f(3) &= 2(3)\sqrt{9-3^2} = 0
\end{aligned}$$

We can clearly see that $f(x)$ attains an absolute maximum of 9 at $x = \frac{3}{\sqrt{2}}$.

Math 180, Exam 2, Spring 2011
Problem 4 Solution

4. Evaluate the following limits:

(a) $\displaystyle\lim_{x\to 0}\left(\frac{1}{x}-\cot x\right)$

(b) $\lim_{x\to 0}(\cot x)(x^2+5x)$

(c) $\lim_{x\to\infty}\dfrac{5x^2-4}{3x^2+7x}$

Solution:

(a) We evaluate the limit by first recognizing that $\cot x = \dfrac{\cos x}{\sin x}$. The limit then becomes:

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{x}-\cot x\right) &= \lim_{x\to 0}\left(\frac{1}{x}-\frac{\cos x}{\sin x}\right)\\ &= \lim_{x\to 0}\frac{\sin x - x\cos x}{x\sin x}\end{aligned}$$

This limit is of the form $\frac{0}{0}$ which is indeterminate. We resolve the indeterminacy using L'Hôpital's Rule.

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x - x\cos x}{x\sin x} &\overset{\text{L'H}}{=} \lim_{x\to 0}\frac{(\sin x - x\cos x)'}{(x\sin x)'}\\ &= \lim_{x\to 0}\frac{\cos x + x\sin x - \cos x}{x\cos x + \sin x}\\ &= \lim_{x\to 0}\frac{x\sin x}{x\cos x + \sin x}\end{aligned}$$

This limit is also of the form $\frac{0}{0}$. We apply L'Hôpital's Rule again.

$$\begin{aligned}\lim_{x\to 0}\frac{x\sin x}{x\cos x + \sin x} &\overset{\text{L'H}}{=} \lim_{x\to 0}\frac{(x\sin x)'}{(x\cos x+\sin x)'}\\ &= \lim_{x\to 0}\frac{x\cos x + \sin x}{-x\sin x + \cos x + \cos x}\\ &= \frac{0\cos 0 + \sin 0}{-0\sin 0 + \cos 0 + \cos 0}\\ &= \boxed{0}\end{aligned}$$

(b) We evaluate the limit by first recognizing that $\cot x = \dfrac{\cos x}{\sin x}$. The limit then becomes:

$$\begin{aligned}
\lim_{x\to 0} (\cot x)(x^2+5x) &= \lim_{x\to 0} \left(\frac{\cos x}{\sin x}\right)(x^2+5x) \\
&= \lim_{x\to 0} \left(\frac{\cos x}{\sin x}\right) x(x+5) \\
&= \lim_{x\to 0} \left(\frac{x}{\sin x}\right)(\cos x)(x+5) \\
&= \left(\lim_{x\to 0} \frac{x}{\sin x}\right)\left(\lim_{x\to 0} (\cos x)(x+5)\right) \\
&= (1)(\cos 0)(0+5) \\
&= \boxed{5}
\end{aligned}$$

(c) In this limit we have a rational function where the numerator and denominator have the same degree. Therefore, the limit is the ratio of the leading coefficients.

$$\lim_{x\to\infty} \frac{5x^2-4}{3x^2+7x} = \boxed{\frac{5}{3}}$$

Math 180, Exam 2, Spring 2011
Problem 5 Solution

5.

(a) Use the tangent line approximation for the function $f(x) = \sqrt{x}$ at the point $x = 9$ to estimate the number $\sqrt{8}$.

(b) Use two steps of Newton's method beginning with $x_1 = 3$ to estimate $\sqrt{8}$, the positive root of $x^2 - 8$.

Solution:

(a) The linearization at $x = 9$ is:

$$L(x) = f(9) + f'(9)(x - 9)$$

The derivative is $f'(x) = \frac{1}{2\sqrt{x}}$. Therefore, the linearization is:

$$L(x) = \sqrt{9} + \frac{1}{2\sqrt{9}}(x - 9)$$

$$L(x) = 3 + \frac{1}{6}(x - 9)$$

The approximate value of $\sqrt{8}$ is the value of $L(8)$ which is:

$$\sqrt{8} \approx L(8) = 3 + \frac{1}{6}(8 - 9) = \boxed{\frac{17}{6}}$$

(b) The function is $f(x) = x^2 - 8$ and its derivative is $f'(x) = 2x$. Therefore, Newton's Method formula is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^2 - 8}{2x_n}$$

The value of x_1 is:

$$x_1 = x_0 - \frac{x_0^2 - 8}{2x_0} = 3 - \frac{3^2 - 8}{2(3)} = \frac{17}{6}$$

The value of x_2 is:

$$x_2 = x_1 - \frac{x_1^2 - 8}{2x_1} = \frac{17}{6} - \frac{(\frac{17}{6})^2 - 8}{2(\frac{17}{6})} = \boxed{\frac{577}{204}}$$