Math 180, Exam 2, Spring 2013
Problem 1 Solution

1. Find the derivative of each function below. **You do not need to simplify your answers.**

(a) $\tan^{-1}(1+\cos x)$

(b) $x^{1/x}$ (logarithmic differentiation may be useful)

(c) $x^3 + y^3 = 3y$ (here y is an implicit function of x).

Solution:

(a) We must use the Chain Rule and the fact that

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{x^2+1}$$

This gives us

$$f'(x) = \frac{1}{(1+\cos(x))^2+1} \cdot \frac{d}{dx}(1+\cos(x))$$

$$f'(x) = \frac{1}{(1+\cos(x))^2+1} \cdot (-\sin(x))$$

(b) We begin by letting $y = x^{1/x}$. Then taking the natural logarithm of both sides of this equation and using the fact that $\ln(a^n) = n\ln(a)$ we obtain

$$\ln(y) = \ln\left(x^{1/x}\right)$$

$$\ln(y) = \frac{1}{x}\ln(x)$$

$$\ln(y) = \frac{\ln(x)}{x}$$

Next we use implicit differentiation to obtain an equation involving $\frac{dy}{dx}$. In order to do this, we must use the Chain and Quotient Rules.

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}\left(\frac{\ln(x)}{x}\right)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{x\cdot\frac{d}{dx}\ln(x) - \ln(x)\cdot\frac{d}{dx}x}{x^2}$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{x\cdot\frac{1}{x} - \ln(x)\cdot 1}{x^2}$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1-\ln(x)}{x^2}$$

Finally, we use algebra to find $\frac{dy}{dx}$ and then write our answer in terms of x only.

$$\frac{dy}{dx} = y \cdot \frac{1-\ln(x)}{x^2}$$

$$\boxed{\frac{dy}{dx} = x^{1/x} \cdot \frac{1-\ln(x)}{x^2}}$$

(c) Here we use implicit differentiation. Using the Power and Chain Rules we obtain

$$\frac{d}{dx}x^3 + \frac{d}{dx}y^3 = \frac{d}{dx}3y$$

$$\boxed{3x^2 + 3y^2 \cdot \frac{dy}{dx} = 3\frac{dy}{dx}}$$

Now we use algebra to find $\frac{dy}{dx}$.

$$\begin{aligned} 3y^2 \cdot \frac{dy}{dx} - 3\frac{dy}{dx} &= -3x^2 \\ \frac{dy}{dx}\left(3y^2 - 3\right) &= -3x^2 \\ \frac{dy}{dx} &= \frac{-3x^2}{3y^2-3} \\ \Aboxed{\frac{dy}{dx} &= \frac{x^2}{1-y^2}} \end{aligned}$$

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Problem 2 Solution

2. For the function given by $f(x) = x^4 - \frac{4}{3}x^3 + 1$ answer the following questions:

(a) Find the interval(s) where $f(x)$ is increasing, decreasing.

(b) Identify all local extrema.

(c) Find the interval(s) where $f(x)$ is concave up, concave down.

(d) Identify all inflection points.

(e) Sketch a graph of f consistent with the information determined in parts (a) and (b). **Your graph does not have to be precise.**

Solution:

(a) To find the intervals of monotonicity we begin by finding the critical points of f. Since f is a polynomial these points will occur whenever $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ 4x^3 - 4x^2 &= 0 \\ 4x^2(x-1) &= 0 \\ \boxed{x = 0,\ x = 1} \end{aligned}$$

We now split the domain of f into the intervals $(-\infty, 0)$, $(0, 1)$, $(1, \infty)$ and let $c = -1, \frac{1}{2}, 2$ be test points in each interval, respectively. We then evaluate $f'(c)$ to determine if f is increasing or decreasing on each interval. Our results are summarized below.

Interval	**Test Number, c**	$f'(c)$	**Sign of $f'(c)$**	**Conclusion**
$(-\infty, 0)$	-1	-8	$-$	decreasing
$(0, 1)$	$\frac{1}{2}$	$-\frac{1}{2}$	$-$	decreasing
$(1, \infty)$	2	16	$+$	increasing

(b) From the table above we see that, although $f'(0) = 0$, there is no sign change in f' across $x = 0$. Thus, $f(0)$ is **neither** a local minimum nor a local maximum. On the other hand, $f'(1) = 0$ and f' changes sign from $-$ to $+$ across $x = 1$ which means that $f(1) = \frac{2}{3}$ is a **local minimum** of f according to the First Derivative Test.

(c) To find the intervals of concavity we begin by finding the *possible* inflection points of f. Since f is a polynomial these points will occur whenever $f''(x) = 0$.

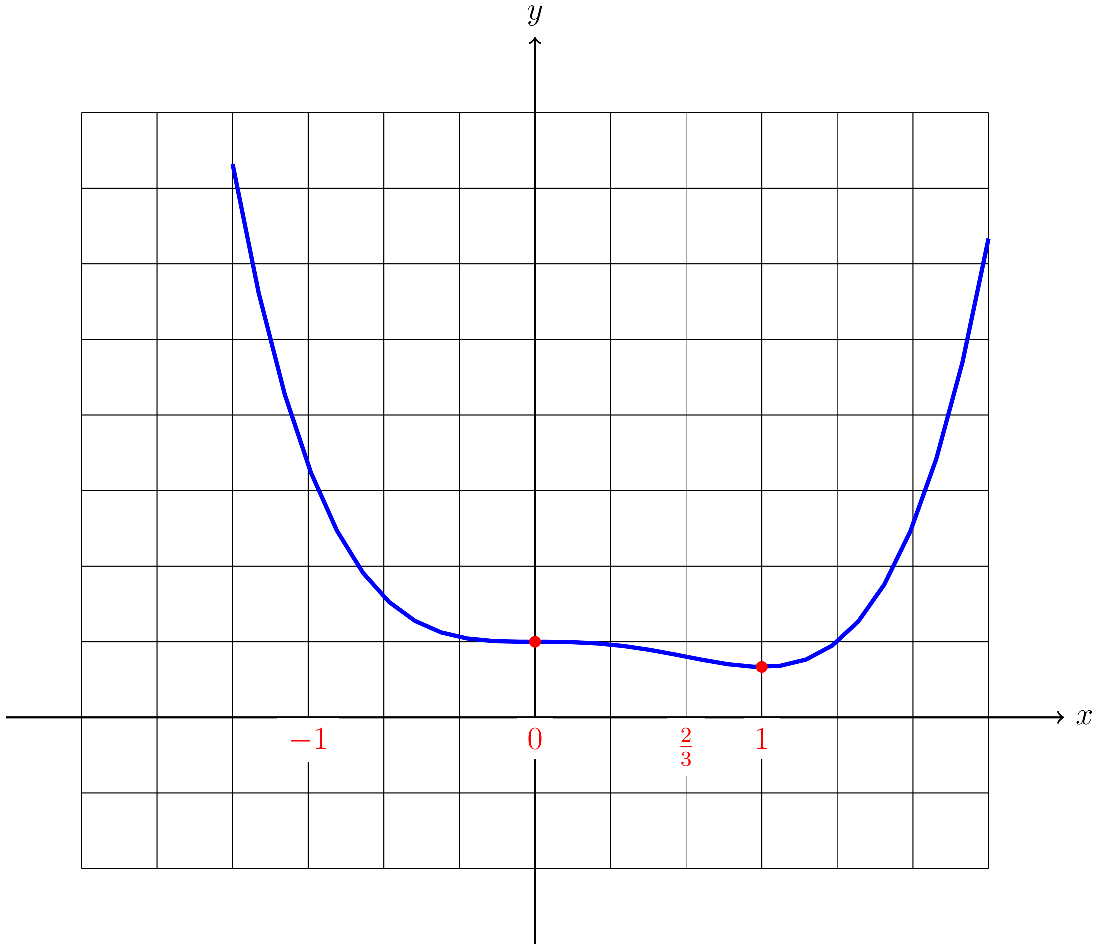
$$\begin{aligned} f''(x) &= 0 \\ 12x^2 - 8x &= 0 \\ 4x(3x-2) &= 0 \end{aligned}$$

$$\boxed{x = 0,\ x = \frac{2}{3}}$$

We now split the domain of f into the intervals $(-\infty, 0)$, $(0, \frac{2}{3})$, $(\frac{2}{3}, \infty)$ and let $c = -1, \frac{1}{2}, 1$ be test points in each interval, respectively. We then evaluate $f''(c)$ to determine if f is concave up or concave down on each interval. Our results are summarized below.

Interval	**Test Number,** c	$f''(c)$	**Sign of** $f''(c)$	**Conclusion**
$(-\infty, 0)$	-1	20	$+$	concave up
$(0, \frac{2}{3})$	$\frac{1}{2}$	-1	$-$	concave down
$(\frac{2}{3}, \infty)$	1	4	$+$	concave up

(d) Since (1) $f''(0) = 0$ and $f''(\frac{2}{3}) = 0$ and (2) f'' changes sign across both $x = 0$ and $x = \frac{2}{3}$, we say that $x = 0, \frac{2}{3}$ are inflection points.



(e)

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Problem 3 Solution

3. Use a linear approximation to estimate the following quantities. In each case indicate whether your answer is an underestimate or an overestimate.

(a) $\ln(0.98)$

(b) $\sin(0.02)$

Solution:

(a) Let $f(x) = \ln(x)$ and $a = 1$ (this is the number closest to 0.98 for which $f(a)$ is an integer.) The linearization of f at $x = a$ is obtained via the formula

$$\boxed{L(x) = f(a) + f'(a)(x - a)}$$

We know that $f(1) = \ln(1) = 0$, by definition. Furthermore, since $f'(x) = \frac{1}{x}$ we know that $f'(1) = 1$. Thus, the function $L(x)$ is

$$\boxed{L(x) = 0 + 1 \cdot (x - 1) = x - 1}$$

The approximate value of $\ln(0.98)$ is then

$$\ln(0.98) \approx L(0.98)$$
$$\ln(0.98) \approx 0.98 - 1$$
$$\boxed{\ln(0.98) \approx -0.02}$$

The function $f(x)$ is concave down for all $x > 0$. This is apparent because $f''(x) = -\frac{1}{x^2} < 0$ for all $x > 0$. Therefore, we know that the graph of the tangent line, $y = x - 1$, will lie above the graph of $y = \ln(x)$ at $x = 0.98$. Thus, our estimate is an **overestimate**.

(b) Let $f(x) = \sin(x)$ and $a = 0$ (this is the number closest to 0.02 for which $f(a)$ is an integer.) We know that $f(0) = \sin(0) = 0$, by definition. Furthermore, since $f'(x) = \cos(x)$ we know that $f'(0) = \cos(0) = 1$. Thus, the function $L(x)$ is

$$\boxed{L(x) = 0 + 1 \cdot (x - 0) = x}$$

The approximate value of $\sin(0.02)$ is then

$$\sin(0.02) \approx L(0.02)$$
$$\boxed{\sin(0.02) \approx 0.02}$$

The function $f(x)$ is concave down on the interval $(0, \pi)$. This is apparent because $f''(x) = -\sin(x) < 0$ for all x in $(0, \pi)$. Therefore, we know that the graph of the tangent line, $y = x$, will lie above the graph of $y = \sin(x)$ at $x = 0.02$. Thus, our estimate is an **overestimate**.

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Problem 4 Solution

4. A rectangle has dimensions 3cm by 2cm. The sides begin increasing in length at a constant rate of 2cm/s. At what rate is the area of the rectangle increasing after 10s?

Solution: Let x and y be the sides of the rectangle. The area of the rectangle is then

$$A = x \cdot y$$

Differentiating each side with respect to t we obtain

$$\frac{d}{dt}A = \frac{d}{dt}\Big(x \cdot y\Big)$$

$$\boxed{\frac{dA}{dt} = \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt}}$$

The rate of change of the length of each side of the triangle is 2 cm/s. Thus,

$$\frac{dA}{dt} = 2(y + x)$$

In order to find the value of $\frac{dA}{dt}$ after 10s, we need to know the lengths of the sides of the rectangle after 10s. Since the sides increase at a constant rate, we know that the size of the rectangle is $(3 + 2 \cdot 10)$ cm by $(2 + 2 \cdot 10)$ cm after 10 s. That is, the size is 23 cm by 22 cm. Therefore, the rate of change of the area of the rectangle is

$$\boxed{\frac{dA}{dt} = 2(23 + 22) = 90}$$

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Problem 5 Solution

5. Find the point on the graph of $y = \frac{2}{x}$, $x > 0$ that is closest to the origin. Hint: Use the square of the distance between $(0,0)$ and $(x, \frac{2}{x})$ as the function to be minimized.

Solution:

Using the hint, we define the function to be minimized as

$$f(x) = x^2 + y^2$$

The constraint in the problem is the equation for the curve. That is

$$y = \frac{2}{x}$$

Plugging this equation into the equation above gives the function

$$f(x) = x^2 + \frac{4}{x^2}$$

whose domain is $x > 0$. The critical points of f are points where $f'(x) = 0$.

$$f'(x) = 0$$
$$2x - \frac{8}{x^3} = 0$$
$$x^4 = 4$$
$$x = \sqrt{2}$$

The corresponding y-coordinate is $y = \frac{2}{\sqrt{2}} = \sqrt{2}$. This point corresponds to an absolute minimum value of f because it is the only critical point and as one moves away from this point along the curve, the distance increases.