# Math 180, Exam 2, Study Guide
## Problem 1 Solution

1. Let $f(x) = \dfrac{x}{x^2+1}$.

- Determine the intervals on which $f$ is increasing and those on which it is decreasing.
- Determine the intervals on which $f$ is concave up and those on which it is concave down.
- Find the critical points of $f$ and determine if they correspond to local extrema.
- Find the asymptotes of $f$.
- Determine the global extrema of $f$.
- Sketch the graph of $f$.

**Solution**: First, we extract as much information as we can from $f'(x)$. We'll start by computing $f'(x)$ using the Quotient Rule.

$$\begin{aligned} f'(x) &= \frac{(x^2+1)(x)' - (x)(x^2+1)'}{(x^2+1)^2} \\ f'(x) &= \frac{(x^2+1)(1) - (x)(2x)}{(x^2+1)^2} \\ f'(x) &= \frac{1-x^2}{(x^2+1)^2} \end{aligned}$$

The critical points of $f(x)$ are the values of $x$ for which either $f'(x)$ does not exist or $f'(x) = 0$. $f'(x)$ is a rational function but the denominator is never 0 so $f'(x)$ exists for all $x \in \mathbb{R}$. Therefore, the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \frac{1-x^2}{(x^2+1)^2} &= 0 \\ 1 - x^2 &= 0 \\ x &= \pm 1 \end{aligned}$$

Thus, $\boxed{x=-1}$ and $\boxed{x=1}$ are the critical points of $f$.

The domain of $f$ is $(-\infty, \infty)$. We now split the domain into the three intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. We then evaluate $f'(x)$ at a test point in each interval to determine the intervals of monotonicity.

| **Interval** | **Test Point**, $c$ | $f'(c)$ | **Sign of** $f'(c)$ |
|---|---|---|---|
| $(-\infty, -1)$ | $-2$ | $f'(-2) = -\frac{3}{25}$ | $-$ |
| $(-1, 1)$ | $0$ | $f'(0) = 1$ | $+$ |
| $(1, \infty)$ | $2$ | $f'(2) = -\frac{3}{25}$ | $-$ |

Using the table, we conclude that $f$ is increasing on $(-1, 1)$ because $f'(x) > 0$ for all $x \in (-1, 1)$ and $f$ is decreasing on $(-\infty, -1) \cup (1, \infty)$ because $f'(x) < 0$ for all $x \in (-\infty, -1) \cup (1, \infty)$. Furthermore, since $f'$ changes sign from $-$ to $+$ at $x = -1$ the First Derivative Test implies that $f(-1) = -\frac{1}{2}$ is a local minimum and since $f'$ changes sign from $+$ to $-$ at $x = 1$ the First Derivative Test implies that $f(1) = \frac{1}{2}$ is a local maximum.

We then extract as much information as we can from $f''(x)$. We'll start by computing $f''(x)$ using the Quotient and Chain Rules.

$$f''(x) = \frac{(x^2+1)^2(1-x^2)' - (1-x^2)[(x^2+1)^2]'}{(x^2+1)^4}$$

$$f''(x) = \frac{(x^2+1)^2(-2x) - (1-x^2)[2(x^2+1)(x^2+1)']}{(x^2+1)^4}$$

$$f''(x) = \frac{-2x(x^2+1)^2 - 2(1-x^2)(x^2+1)(2x)}{(x^2+1)^4}$$

$$f''(x) = \frac{-2x(x^2+1) - 4x(1-x^2)}{(x^2+1)^3}$$

$$f''(x) = \frac{2x^3 - 6x}{(x^2+1)^3}$$

The possible inflection points of $f(x)$ are the values of $x$ for which either $f''(x)$ does not exist or $f''(x) = 0$. Since $f''(x)$ exists for all $x \in \mathbb{R}$, the only possible inflection points are solutions to $f''(x) = 0$.

$$f''(x) = 0$$

$$\frac{2x^3 - 6x}{(x^2+1)^3} = 0$$

$$2x^3 - 6x = 0$$

$$2x(x^2 - 3) = 0$$

$$x = 0, \; x = \pm\sqrt{3}$$

We now split the domain into the four intervals $(-\infty, -\sqrt{3})$, $(-\sqrt{3}, 0)$, $(0, \sqrt{3})$, and $(\sqrt{3}, \infty)$. We then evaluate $f''(x)$ at a test point in each interval to determine the intervals of concavity.

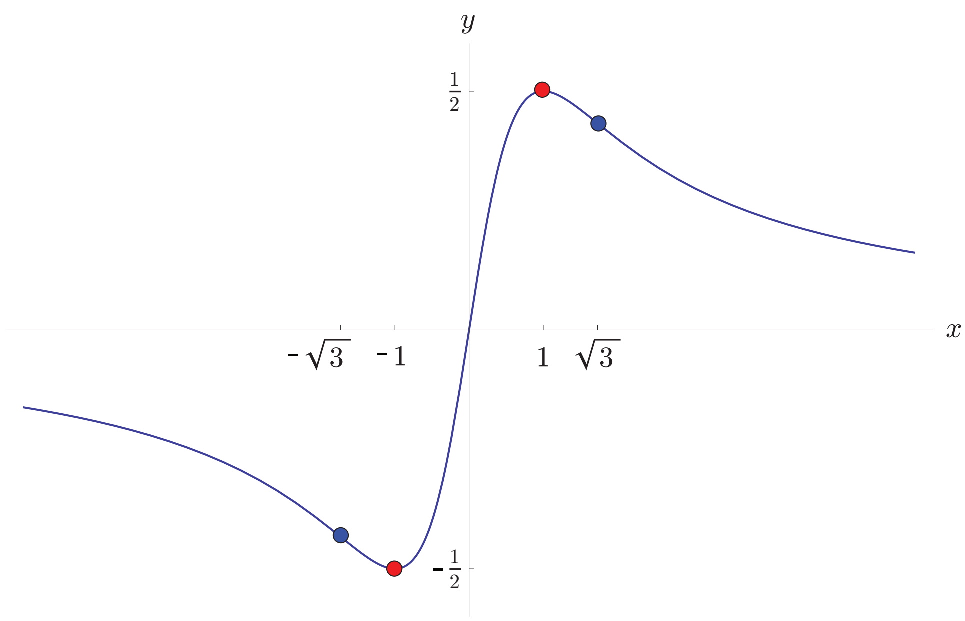
| **Interval** | **Test Point**, $c$ | $f'(c)$ | **Sign of** $f'(c)$ |
|---|---|---|---|
| $(-\infty, -\sqrt{3})$ | $-2$ | $f''(-2) = -\frac{4}{125}$ | $-$ |
| $(-\sqrt{3}, 0)$ | $-1$ | $f''(-1) = \frac{1}{2}$ | $+$ |
| $(0, \sqrt{3})$ | $1$ | $f''(1) = -\frac{1}{2}$ | $-$ |
| $(\sqrt{3}, \infty)$ | $2$ | $f''(2) = \frac{4}{125}$ | $+$ |

Using the table, we conclude that $f$ is concave up on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$ because $f''(x) > 0$ for all $x \in (-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$ and $f$ is concave down on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$ because $f''(x) < 0$ for all $x \in (-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$. Furthermore, since $f''$ changes sign at $x = -\sqrt{3}$, $x = 0$, and $x = \sqrt{3}$, all three points are inflection points.

$f(x)$ does not have a vertical asymptote because it is continuous for all $x \in \mathbb{R}$. The horizontal asymptote is $y = 0$ because

$$\begin{aligned} \lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{x}{x^2+1} \\ &= \lim_{x\to\infty} \frac{x}{x^2} \\ &= \lim_{x\to\infty} \frac{1}{x} \\ &= 0 \end{aligned}$$

The absolute minimum of $f(x)$ is $\boxed{-\frac{1}{2}}$ at $x = -1$ and the absolute maximum is $\boxed{\frac{1}{2}}$ at $x = 1$.



**Math 180, Exam 2, Study Guide**
**Problem 2 Solution**

2. Let $f(x) = xe^x$.

   i) Find and classify the critical points of $f$.

   ii) Find the global minimum of $f$ over the entire real line.

**Solution**:

i) The critical points of $f(x)$ are the values of $x$ for which either $f'(x) = 0$ or $f'(x)$ does not exist. Since $f(x)$ is a product of two infinitely differentiable functions, we know that $f'(x)$ exists for all $x \in \mathbb{R}$. Therefore, the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ (xe^x)' &= 0 \\ (x)(e^x)' + (e^x)(x)' &= 0 \\ xe^x + e^x &= 0 \\ e^x(x+1) &= 0 \\ x &= -1 \end{aligned}$$

$\boxed{x = -1}$ is the only critical point because $e^x > 0$ for all $x \in \mathbb{R}$.

We use the First Derivative Test to classify the critical point $x = -1$. The domain of $f$ is $(-\infty, \infty)$. Therefore, we divide the domain into the two intervals $(-\infty, -1)$ and $(-1, \infty)$. We then evaluate $f'(x)$ at a test point in each interval to determine where $f'(x)$ is positive and negative.

| **Interval** | **Test Number,** $c$ | $f'(c)$ | **Sign of** $f'(c)$ |
|---|---|---|---|
| $(-\infty, -1)$ | $-2$ | $-e^{-2}$ | $-$ |
| $(-1, \infty)$ | $0$ | $1$ | $+$ |

Since $f$ changes sign from $-$ to $+$ at $x = -1$ the First Derivative Test implies that $f(-1) = -e^{-1}$ is a local minimum.

ii) From the table in part (a), we conclude that $f$ is decreasing on the interval $(-\infty, -1)$ and increasing on the interval $(-1, \infty)$. Therefore, $\boxed{f(-1) = -e^{-1}}$ is the global minimum of $f$ over the entire real line.

**Math 180, Exam 2, Study Guide**
**Problem 3 Solution**

3. Find the minimum and maximum of the function $f(x) = \sqrt{6x - x^3}$ over the interval $[0, 2]$.

**Solution**: The minimum and maximum values of $f(x)$ will occur at a critical point in the interval $[0, 2]$ or at one of the endpoints. The critical points are the values of $x$ for which either $f'(x) = 0$ or $f'(x)$ does not exist. The derivative $f'(x)$ is found using the Chain Rule.

$$\begin{aligned}
f'(x) &= \left[\left(6x - x^3\right)^{1/2}\right]' \\
f'(x) &= \frac{1}{2}\left[\left(6x - x^3\right)^{-1/2}\right] \cdot \left(6x - x^3\right)' \\
f'(x) &= \frac{1}{2}\left[\left(6x - x^3\right)^{-1/2}\right] \cdot \left(6 - 3x^2\right) \\
f'(x) &= \frac{6 - 3x^2}{2\sqrt{6x - x^3}}
\end{aligned}$$

$f'(x)$ does not exist when the denominator is 0. This will happen when $6x - x^3 = 0$. The solutions to this equation are obtained as follows:

$$\begin{aligned}
6x - x^3 &= 0 \\
x(6 - x^2) &= 0 \\
x = 0,\ x &= \pm\sqrt{6}
\end{aligned}$$

The critical point $x = 0$ is an endpoint of $[0, 2]$. The critical points $x = \pm\sqrt{6}$ both lie outside $[0, 2]$. Therefore, there are no critical points in $[0, 2]$ where $f'(x)$ does not exist.

The only critical points are points where $f'(x) = 0$.

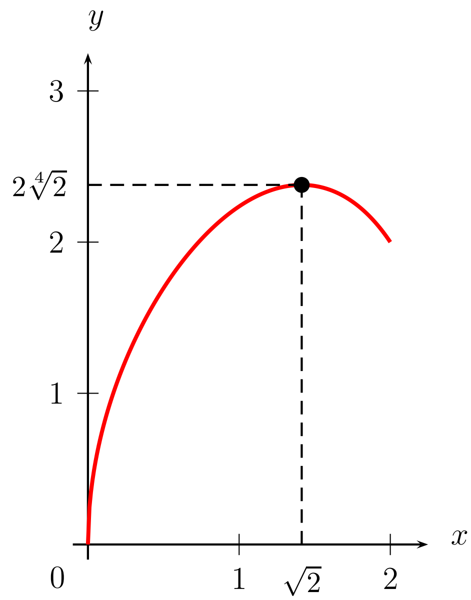
$$\begin{aligned}
f'(x) &= 0 \\
\frac{6 - 3x^2}{2\sqrt{6x - x^3}} &= 0 \\
6 - 3x^2 &= 0 \\
x^2 &= 2 \\
x &= \pm\sqrt{2}
\end{aligned}$$

The critical point $x = -\sqrt{2}$ lies outside $[0, 2]$. Therefore, $x = \sqrt{2}$ is the only critical point in $[0, 2]$ where $f'(x) = 0$.

We now evaluate $f(x)$ at $x = 0,\ \sqrt{2}$, and 2.

$$\begin{aligned}
f(0) &= \sqrt{6(0) - 0^3} = 0 \\
f\left(\sqrt{2}\right) &= \sqrt{6\left(\sqrt{2}\right) - \left(\sqrt{2}\right)^3} = 2\sqrt[4]{2} \\
f(2) &= \sqrt{6(2) - 2^3} = 2
\end{aligned}$$

The minimum value of $f(x)$ on $[0, 2]$ is $\boxed{0}$ because it is the smallest of the above values of $f$. The maximum is $\boxed{2\sqrt[4]{2}}$ because it is the largest.



**Math 180, Exam 2, Study Guide**
**Problem 4 Solution**

4. Let $f(x) = 3x - x^3$.

i) On what interval(s) is $f$ increasing?

ii) On what interval(s) is $f$ decreasing?

iii) On what interval(s) is $f$ concave up?

iv) On what interval(s) is $f$ concave down?

v) Sketch the graph of $f$.

**Solution**:

i) We begin by finding the critical points of $f(x)$. The critical points of $f(x)$ are the values of $x$ for which either $f'(x)$ does not exist or $f'(x) = 0$. Since $f(x)$ is a polynomial, $f'(x)$ exists for all $x \in \mathbb{R}$ so the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(3x - x^3\right)' &= 0 \\ 3 - 3x^2 &= 0 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

The domain of $f$ is $(-\infty, \infty)$. We now split the domain into the three intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. We then evaluate $f'(x)$ at a test point in each interval to determine the intervals of monotonicity.

| **Interval** | **Test Point**, $c$ | $f'(c)$ | **Sign of** $f'(c)$ |
|---|---|---|---|
| $(-\infty, -1)$ | $-2$ | $f'(-2) = -9$ | $-$ |
| $(-1, 1)$ | $0$ | $f'(0) = 3$ | $+$ |
| $(1, \infty)$ | $2$ | $f'(2) = -9$ | $-$ |

Using the table we conclude that $f$ is increasing on $\boxed{(-1, 1)}$ because $f'(x) > 0$ for all $x \in (-1, 1)$

ii) From the table above we conclude that $f$ is decreasing on $\boxed{(-\infty, -1) \cup (1, \infty)}$ because $f'(x) < 0$ for all $x \in (-\infty, -1) \cup (1, \infty)$.

iii) To determine the intervals of concavity we start by finding solutions to the equation $f''(x) = 0$ and where $f''(x)$ does not exist. However, since $f(x)$ is a polynomial we know that $f''(x)$ will exist for all $x \in \mathbb{R}$. The solutions to $f''(x) = 0$ are:

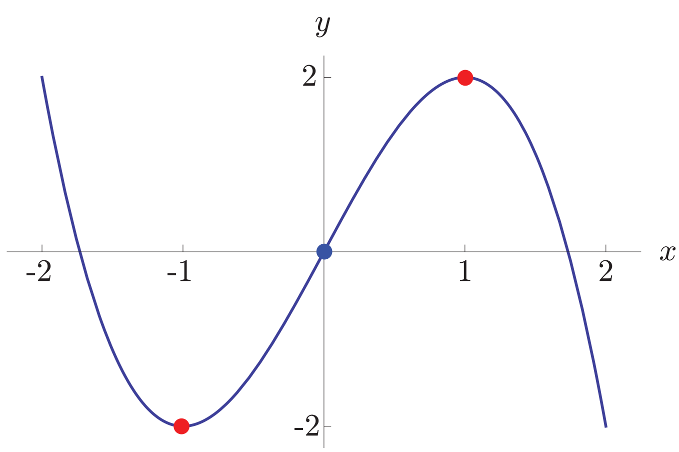
$$\begin{aligned} f''(x) &= 0 \\ -6x &= 0 \\ x &= 0 \end{aligned}$$

We now split the domain into the two intervals $(-\infty, 0)$ and $(0, \infty)$. We then evaluate $f''(x)$ at a test point in each interval to determine the intervals of concavity.

| **Interval** | **Test Point**, $c$ | $f''(c)$ | **Sign of** $f''(c)$ |
|---|---|---|---|
| $(-\infty, 0)$ | $-1$ | $f''(-1) = 6$ | $+$ |
| $(0, \infty)$ | $1$ | $f''(0) = -6$ | $-$ |

Using the table we conclude that $f$ is concave up on $\boxed{(-\infty, 0)}$ because $f''(x) > 0$ for all $x \in (-\infty, 0)$.

iv) From the above table we conclude that $f$ is concave down on $\boxed{(0, \infty)}$ because $f''(x) < 0$ for all $x \in (0, \infty)$.




v)

**Math 180, Exam 2, Study Guide**
**Problem 5 Solution**

5. For a function $f(x)$ we know that $f(3) = 2$ and that $f'(3) = -3$. Give an estimate for $f(2.91)$.

**Solution**: We will estimate $f(2.91)$ using $L(2.91)$, the linearization $L(x)$ of the function $f(x)$ at $a = 3$ evaluated at $x = 2.91$. The function $L(x)$ is defined as:

$$L(x) = f(3) + f'(3)(x - 3)$$

Using $f(3) = 2$ and $f'(3) = -3$ we have:

$$L(x) = 2 - 3(x - 3)$$

Plugging $x = 2.91$ into $L(x)$ we get:

$$L(2.91) = 2 - 3(2.91 - 3)$$
$$L(2.91) = 2.27$$

Therefore, $\boxed{f(2.91) \approx L(2.91) = 2.27}$.

## Math 180, Exam 2, Study Guide
## Problem 6 Solution

6. Let $f(x) = \dfrac{x^2+1}{x+1}$. Find the best linear approximation of $f$ around the point $x = 0$ and use it in order to estimate $f(0.2)$. Would this be an underestimate or an overestimate?

**Solution**: The linearization $L(x)$ of $f(x)$ at $x = 0$ is defined as:

$$L(x) = f(0) + f'(0)(x-0)$$

The derivative $f'(x)$ is found using the Quotient Rule:

$$\begin{aligned} f'(x) &= \left(\frac{x^2+1}{x+1}\right)' \\ &= \frac{(x+1)(x^2+1)' - (x^2+1)(x+1)'}{(x+1)^2} \\ &= \frac{(x+1)(2x) - (x^2+1)(1)}{(x+1)^2} \\ &= \frac{x^2+2x-1}{(x+1)^2} \end{aligned}$$

At $x = 0$, the values of $f'$ and $f$ are:

$$\begin{aligned} f'(0) &= \frac{0^2 + 2(0) - 1}{(0+1)^2} = -1 \\ f(0) &= \frac{0^2+1}{0+1} = 1 \end{aligned}$$

The linearization $L(x)$ is then:

$$\boxed{L(x) = 1 - x}$$

Since $f(0.2) \approx L(0.2)$ we find that:

$$\begin{aligned} f(0.2) &\approx L(0.2) \\ &\approx 1 - 0.2 \\ &\approx \boxed{0.8} \end{aligned}$$

The actual value of $f(0.2)$ is:

$$f(0.2) = \frac{0.2^2+1}{0.2+1} = \frac{1.04}{1.2} = \frac{13}{15} > \frac{12}{15} = 0.8$$

So $L(0.2) = 0.8$ is an **underestimate**.

**Math 180, Exam 2, Study Guide**
**Problem 7 Solution**

7. A rectangular farm of total area 20,000 sq. feet is to be fenced on three sides. Find the dimensions that are going to give the minimum cost.

**Solution**: We begin by letting $x$ be the length of one side, $y$ be the lengths of the remaining two fenced sides, and $C > 0$ be the cost of the fence per unit length. The function we seek to minimize is the cost of the fencing:

$$\textbf{Function}: \quad \text{Cost} = C(x + 2y) \tag{1}$$

The constraint in this problem is that the area of the garden is 20,000 square meters.

$$\textbf{Constraint}: \quad xy = 20,000 \tag{2}$$

Solving the constraint equation (2) for $y$ we get:

$$y = \frac{20,000}{x} \tag{3}$$

Plugging this into the function (1) and simplifying we get:

$$\begin{aligned} \text{Cost} &= C\left[x + 2\left(\frac{20,000}{x}\right)\right] \\ f(x) &= C\left(x + \frac{40,000}{x}\right) \end{aligned}$$

We want to find the absolute minimum of $f(x)$ on the **interval** $(0, \infty)$. We choose this interval because $x$ must be nonnegative ($x$ represents a length) and non-zero (if $x$ were 0, then the area would be 0 but it must be $20,000$).

The absolute minimum of $f(x)$ will occur either at a critical point of $f(x)$ in $(0, \infty)$ or it will not exist because the interval is open. The critical points of $f(x)$ are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ C\left(x + \frac{40,000}{x}\right)' &= 0 \\ C\left(1 - \frac{40,000}{x^2}\right) &= 0 \\ x^2 &= 40,000 \\ x &= \pm 200 \end{aligned}$$

However, since $x = -200$ is outside $(0, \infty)$, the only critical point is $x = 200$. Plugging this into $f(x)$ we get:

$$f(200) = C\left(200 + \frac{40,000}{200}\right) = 400C$$

Taking the limits of $f(x)$ as $x$ approaches the endpoints we get:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} C\left(x + \frac{40,000}{x}\right) = C\,(0+\infty) = \infty$$
$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} C\left(x + \frac{40,000}{x}\right) = C\,(\infty+0) = \infty$$

both of which are larger than $400C$. We conclude that the cost is an absolute minimum at $\boxed{x = 200}$ and that the resulting cost is $400C$. The last step is to find the corresponding value for $y$ by plugging $x = 200$ into equation (3).

$$y = \frac{20,000}{x} = \frac{20,000}{200} = \boxed{100}$$

**Math 180, Exam 2, Study Guide**
**Problem 8 Solution**

8. Let $f(x) = 3x^5 - x^3$.

- Find the critical points of $f$.
- Determine the intervals on which $f$ is increasing and the ones on which it is decreasing.
- Determine the intervals on which $f$ is concave up and the ones on which it is concave down.
- Determine the inflection points of $f$.
- Sketch the graph of $f$.

**Solution**:

- The critical points of $f(x)$ are the values of $x$ for which either $f'(x)$ does not exist or $f'(x) = 0$. Since $f(x)$ is a polynomial, $f'(x)$ exists for all $x \in \mathbb{R}$ so the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(3x^5 - x^3\right)' &= 0 \\ 15x^4 - 3x^2 &= 0 \\ 3x^2(5x^2 - 1) &= 0 \\ x = 0,\ x &= \pm\frac{1}{\sqrt{5}} \end{aligned}$$

  Therefore, the critical points of $f$ are $\boxed{x = 0, \pm\frac{1}{\sqrt{5}}}$.

- The domain of $f$ is $(-\infty, \infty)$. We now split the domain into the four intervals $(-\infty, -\frac{1}{\sqrt{5}})$, $(-\frac{1}{\sqrt{5}}, 0)$, $(0, \frac{1}{\sqrt{5}})$, and $(\frac{1}{\sqrt{5}}, \infty)$. We then evaluate $f'(x)$ at a test point in each interval to determine the intervals of monotonicity.

| **Interval** | **Test Point**, $c$ | $f'(c)$ | **Sign of** $f'(c)$ |
|---|---|---|---|
| $(-\infty, -\frac{1}{\sqrt{5}})$ | $-1$ | $f'(-1) = 12$ | $+$ |
| $(-\frac{1}{\sqrt{5}}, 0)$ | $-\frac{1}{5}$ | $f'(-\frac{1}{5}) = -\frac{12}{125}$ | $-$ |
| $(0, \frac{1}{\sqrt{5}})$ | $\frac{1}{5}$ | $f'(\frac{1}{5}) = -\frac{12}{125}$ | $-$ |
| $(\frac{1}{\sqrt{5}}, \infty)$ | $1$ | $f'(1) = 12$ | $+$ |

Using the table we conclude that $f$ is increasing on $(-\infty, -\frac{1}{\sqrt{5}}) \cup (\frac{1}{\sqrt{5}}, \infty)$ because $f'(x) > 0$ for all $x \in (-\infty, -\frac{1}{\sqrt{5}}) \cup (\frac{1}{\sqrt{5}}, \infty)$ and $f$ is decreasing on $(-\frac{1}{\sqrt{5}}, 0) \cup (0, \frac{1}{\sqrt{5}})$ because $f'(x) < 0$ for all $x \in (-\frac{1}{\sqrt{5}}, 0) \cup (0, \frac{1}{\sqrt{5}})$.

- To determine the intervals of concavity we start by finding solutions to the equation $f''(x) = 0$ and where $f''(x)$ does not exist. However, since $f(x)$ is a polynomial we know that $f''(x)$ will exist for all $x \in \mathbb{R}$. The solutions to $f''(x) = 0$ are:

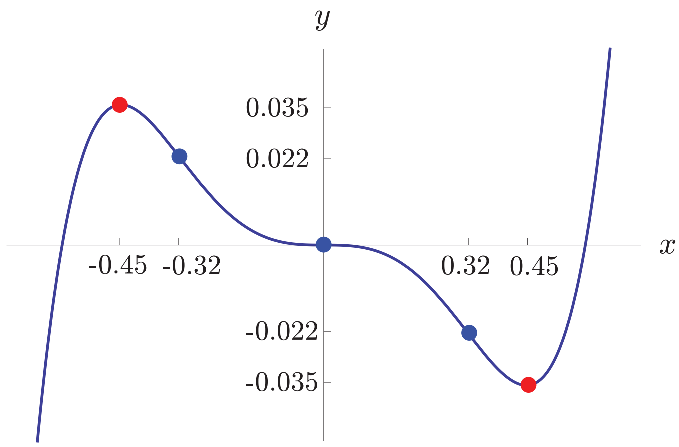
$$\begin{aligned} f''(x) &= 0 \\ \left(15x^4 - 3x^2\right)' &= 0 \\ 60x^3 - 6x &= 0 \\ 6x(10x^2 - 1) &= 0 \\ x = 0,\ x &= \pm\frac{1}{\sqrt{10}} \end{aligned}$$

We now split the domain into the four intervals $(-\infty, -\frac{1}{\sqrt{10}})$, $(-\frac{1}{\sqrt{10}}, 0)$, $(0, \frac{1}{\sqrt{10}})$, and $(\frac{1}{\sqrt{10}}, \infty)$. We then evaluate $f''(x)$ at a test point in each interval to determine the intervals of concavity.

| **Interval** | **Test Point**, $c$ | $f'(c)$ | **Sign of** $f'(c)$ |
|---|---|---|---|
| $(-\infty, -\frac{1}{\sqrt{10}})$ | $-1$ | $f''(-1) = -54$ | $-$ |
| $(-\frac{1}{\sqrt{10}}, 0)$ | $-\frac{1}{10}$ | $f''(-\frac{1}{10}) = \frac{27}{50}$ | $+$ |
| $(0, \frac{1}{\sqrt{10}})$ | $\frac{1}{10}$ | $f''(\frac{1}{10}) = -\frac{27}{50}$ | $-$ |
| $(\frac{1}{\sqrt{10}}, \infty)$ | $1$ | $f''(1) = 54$ | $+$ |

Using the table we conclude that $f$ is concave up on $(-\frac{1}{\sqrt{10}}, 0) \cup (\frac{1}{\sqrt{10}}, \infty)$ because $f''(x) > 0$ for all $x \in (-\frac{1}{\sqrt{10}}, 0) \cup (\frac{1}{\sqrt{10}}, \infty)$ and that $f$ is concave down on $(-\infty, -\frac{1}{\sqrt{10}}) \cup (0, \frac{1}{\sqrt{10}})$ because $f''(x) < 0$ for all $x \in (-\infty, -\frac{1}{\sqrt{10}}) \cup (0, \frac{1}{\sqrt{10}})$.

- An inflection point of $f(x)$ is a point where $f''(x)$ changes sign. From the above table we conclude that $\boxed{x = 0, \pm\frac{1}{\sqrt{10}}}$ are inflection points.


y
0.035
0.022
x
-0.45
-0.32
0.32
0.45
-0.022
-0.035

**Math 180, Exam 2, Study Guide**
**Problem 9 Solution**

9. A rectangle has its left lower corner at $(0,0)$ and its upper right corner on the graph of

$$f(x) = x^2 + \frac{1}{x^2}$$

i) Express its area as a function of $x$.

ii) Determine $x$ for which the area is a minimum.

iii) Can the area of such a rectangle be as large as we please?

**Solution**:

i) The dimensions of the rectangle are $x$ and $y$. Therefore, the area of the rectangle has the equation:

$$\text{Area} = xy \tag{1}$$

We are asked to write the area as a function of $x$ alone. Therefore, we must find an equation that relates $x$ to $y$ so that we can eliminate $y$ from the area equation. This equation is

$$y = x^2 + \frac{1}{x^2} \tag{2}$$

because $(x, y)$ must lie on this curve. Plugging this into the area equation we get:

$$\text{Area} = x\left(x^2 + \frac{1}{x^2}\right)$$

$$\boxed{g(x) = x^3 + \frac{1}{x}}$$

ii) We seek the value of $x$ that minimizes $g(x)$. The interval in the problem is $(0,\infty)$ because the domain of $f(x)$ is $(-\infty,0)\cup(0,\infty)$ but $(x, y)$ must be in the first quadrant.

The absolute minimum of $f(x)$ will occur either at a critical point of $f(x)$ in $(0,\infty)$ or it will not exist because the interval is open. The critical points of $f(x)$ are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(x^3 + \frac{1}{x}\right)' &= 0 \\ 3x^2 - \frac{1}{x^2} &= 0 \\ 3x^4 - 1 &= 0 \\ x &= \pm\frac{1}{\sqrt[4]{3}} \end{aligned}$$

However, since $x = -\frac{1}{\sqrt[4]{3}}$ is outside $(0, \infty)$, the only critical point is $x = \frac{1}{\sqrt[4]{3}}$. Plugging this into $g(x)$ we get:

$$f\left(\frac{1}{\sqrt[4]{3}}\right) = \left(\frac{1}{\sqrt[4]{3}}\right)^3 + \frac{1}{\frac{1}{\sqrt[4]{3}}} = \frac{1}{\sqrt[4]{27}} + \sqrt[4]{3}$$

Taking the limits of $f(x)$ as $x$ approaches the endpoints we get:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(x^3 + \frac{1}{x}\right) = 0 + \infty = \infty$$
$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left(x^3 + \frac{1}{x}\right) = \infty + 0 = \infty$$

both of which are larger than $\frac{1}{\sqrt[4]{27}} + \sqrt[4]{3}$. We conclude that the area is an absolute minimum at $\boxed{x = \frac{1}{\sqrt[4]{3}}}$ and that the resulting area is $\frac{1}{\sqrt[4]{27}} + \sqrt[4]{3}$.

iii) We can make the rectangle as large as we please by taking $x \to 0^+$ or $x \to \infty$.

**Math 180, Exam 2, Study Guide**
**Problem 10 Solution**

10. A box has square base of side $x$ and constant surface area equal to 12 $\text{m}^2$.

i) Express its volume as a function of $x$.

ii) Find the maximum volume of such a box.

**Solution**:

i) We begin by letting $x$ be the length of one side of the base and $y$ be the height of the box. The volume then has the equation:

$$\text{Volume} = x^2 y \tag{1}$$

We are asked to write the volume as a function of width, $x$. Therefore, we must find an equation that relates $x$ to $y$ so that we can eliminate $y$ from the volume equation.

The constraint in the problem is that the total surface area is 12. This gives us the equation

$$2x^2 + 4xy = 12 \tag{2}$$

Solving this equation for $y$ we get

$$\begin{aligned} 2x^2 + 4xy &= 12 \\ x^2 + 2xy &= 6 \\ y &= \frac{6 - x^2}{2x} \end{aligned} \tag{3}$$

We then plug this into the volume equation (1) to write the volume in terms of $x$ only.

$$\begin{aligned} \text{Volume} &= x^2 y \\ \text{Volume} &= x^2 \left( \frac{6 - x^2}{2x} \right) \\ &\boxed{f(x) = 3x - \frac{1}{2}x^3} \end{aligned} \tag{4}$$

ii) We seek the value of $x$ that maximizes $f(x)$. The interval in the problem is $(0, \sqrt{6}]$. We know that $x > 0$ because $x$ must be positive and nonzero (otherwise, the surface area would be 0 and it must be 12). It is possible that $y = 0$ in which case the surface area constraint would give us $2x^2 + 4x(0) = 12 \quad \Rightarrow \quad x^2 = 6 \quad \Rightarrow \quad x = \sqrt{6}$.

The absolute maximum of $f(x)$ will occur either at a critical point of $f(x)$ in $(0, \sqrt{6}]$, at $x = \sqrt{6}$, or it will not exist. The critical points of $f(x)$ are solutions to $f'(x) = 0$.

$$
\begin{aligned}
f'(x) &= 0 \\
\left(3x - \frac{1}{2}x^3\right)' &= 0 \\
3 - \frac{3}{2}x^2 &= 0 \\
x^2 &= 2 \\
x &= \pm\sqrt{2}
\end{aligned}
$$

However, since $x = -\sqrt{2}$ is outside $(0, \sqrt{6}]$, the only critical point is $x = \sqrt{2}$. Plugging this into $f(x)$ we get:

$$
f\left(\sqrt{2}\right) = 3\left(\sqrt{2}\right) - \frac{1}{2}\left(\sqrt{2}\right)^3 = 2\sqrt{2}
$$

Evaluating $f(x)$ at $x = \sqrt{6}$ and taking the limit of $f(x)$ as $x$ approaches $x = 0$ we get:

$$
\begin{aligned}
\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left(3x - \frac{1}{2}x^3\right) &= 0 \\
f\left(\sqrt{6}\right) = 3\left(\sqrt{6}\right) - \frac{1}{2}\left(\sqrt{6}\right)^3 &= 0
\end{aligned}
$$

both of which are smaller than $2\sqrt{2}$. We conclude that the volume is an absolute maximum at $x = \sqrt{2}$ and that the resulting volume is $\boxed{2\sqrt{2}\ \mathrm{m}^3}$.

**Math 180, Exam 2, Study Guide**
**Problem 11 Solution**

11. Use the Newton approximation method in order to find $x_2$ as an estimate for the positive root of the equation $x^2 - 5 = 0$ when $x_0 = 5$.

**Solution**: The Newton's method formula to compute $x_1$ is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

where $f(x) = x^2 - 5$. The derivative $f'(x)$ is $f'(x) = 2x$. Plugging $x_0 = 5$ into the formula we get:

$$x_1 = x_0 - \frac{x_0^2 - 5}{2x_0}$$
$$x_1 = 5 - \frac{5^2 - 5}{2(5)}$$
$$x_1 = 5 - \frac{20}{10}$$
$$x_1 = 3$$

The Newton's method formula to compute $x_2$ is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Plugging $x_1 = 3$ into the formula we get:

$$x_2 = x_1 - \frac{x_1^2 - 5}{2x_1}$$
$$x_2 = 3 - \frac{3^2 - 5}{2(3)}$$
$$x_2 = 3 - \frac{4}{6}$$
$$\boxed{x_2 = \frac{7}{3}}$$

# Math 180, Exam 2, Study Guide
# Problem 12 Solution

12. Use L'Hôpital's Rule in order to compute the following limits:

$$\lim_{x\to 0}\frac{\ln(3x+1)}{\ln(5x+1)} \qquad \lim_{x\to 0^+} x\ln x \qquad \lim_{x\to 0}\frac{e^{3x}-1}{\tan x}$$

$$\lim_{x\to 4}\left(\frac{1}{\sqrt{x}-2}-\frac{4}{x-4}\right) \qquad \lim_{x\to +\infty}\frac{e^x}{x+\ln x}$$

**Solution**: Upon substituting $x=0$ into the function $\frac{\ln(3x+1)}{\ln(5x+1)}$ we get

$$\frac{\ln(3(0)+1)}{\ln(5(0)+1)}=\frac{0}{0}$$

which is indeterminate. We resolve the indeterminacy using L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to 0}\frac{\ln(3x+1)}{\ln(5x+1)} &\overset{\text{L'H}}{=} \lim_{x\to 0}\frac{(\ln(3x+1))'}{(\ln(5x+1))'}\\
&= \lim_{x\to 0}\frac{\frac{1}{3x+1}\cdot 3}{\frac{1}{5x+1}\cdot 5}\\
&= \lim_{x\to 0}\frac{3}{5}\cdot\frac{5x+1}{3x+1}\\
&= \frac{3}{5}\cdot\frac{5(0)+1}{3(0)+1}\\
&= \boxed{\frac{3}{5}}
\end{aligned}$$

As $x\to 0^+$ we find that $x\ln x\to 0\cdot(-\infty)$ which is indeterminate. However, it is not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ which is required to use L'Hôpital's Rule. To get the limit into one of the two required forms, we rewrite $x\ln x$ as follows:

$$x\ln x=\frac{\ln x}{\frac{1}{x}}$$

As $x\to 0^+$, we find that $\frac{\ln x}{1/x}\to\frac{-\infty}{\infty}$. We can now use L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to 0^+} x\ln x &= \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}\\
&\overset{\text{L'H}}{=} \lim_{x\to 0^+}\frac{(\ln x)'}{(\frac{1}{x})'}\\
&= \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}\\
&= \lim_{x\to 0^+} -x\\
&= \boxed{0}
\end{aligned}$$

Upon substituting $x = 0$ into the function $\frac{e^{3x}-1}{\tan x}$ we get

$$\frac{e^{3(0)} - 1}{\tan 0} = \frac{0}{0}$$

which is indeterminate. We resolve the indeterminacy using L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to 0} \frac{e^{3x} - 1}{\tan x} &\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{(e^{3x} - 1)'}{(\tan x)'} \\
&= \lim_{x\to 0} \frac{3e^{3x}}{\sec^2 x} \\
&= \lim_{x\to 0} 3e^{3x} \cos^2 x \\
&= 3e^{3(0)} \cos^2 0 \\
&= \boxed{3}
\end{aligned}$$

Upon substituting $x = 4$ into the function $\frac{1}{\sqrt{x}-2} - \frac{4}{x-4}$ we get

$$\frac{1}{\sqrt{4} - 2} - \frac{4}{4 - 4} = \infty - \infty$$

which is indeterminate. In order to use L'Hôpital's Rule we need the limit to be of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. To get the limit into one of these forms, we rewrite the function as follows:

$$\begin{aligned}
\frac{1}{\sqrt{x} - 2} - \frac{4}{x - 4} &= \frac{x - 4 - 4(\sqrt{x} - 2)}{(\sqrt{x} - 2)(x - 4)} \\
&= \frac{x - 4\sqrt{x} + 4}{(\sqrt{x} - 2)(x - 4)} \\
&= \frac{(\sqrt{x} - 2)(\sqrt{x} - 2)}{(\sqrt{x} - 2)(x - 4)} \\
&= \frac{\sqrt{x} - 2}{x - 4}
\end{aligned}$$

Upon substituting $x = 4$ into the $\frac{\sqrt{x}-2}{x-4}$ we get

$$\frac{\sqrt{4} - 2}{4 - 4} = \frac{0}{0}$$

which is now of the indeterminate form $\frac{0}{0}$. We can now use L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to 4}\left(\frac{1}{\sqrt{x}-2}-\frac{4}{x-4}\right) &= \lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} \\
&\overset{\text{L'H}}{=} \lim_{x\to 4}\frac{(\sqrt{x}-2)'}{(x-4)'} \\
&= \lim_{x\to 4}\frac{\frac{1}{2\sqrt{x}}}{1} \\
&= \frac{1}{2\sqrt{4}} \\
&= \boxed{\frac{1}{4}}
\end{aligned}$$

As $x \to +\infty$, we find that $\frac{e^x}{x+\ln x} \to \frac{\infty}{\infty}$ which is indeterminate. We resolve the indeterminacy using L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to +\infty}\frac{e^x}{x+\ln x} &\overset{\text{L'H}}{=} \lim_{x\to +\infty}\frac{(e^x)'}{(x+\ln x)'} \\
&= \lim_{x\to +\infty}\frac{e^x}{1+\frac{1}{x}} \\
&= \frac{+\infty}{1+0} \\
&= \boxed{+\infty}
\end{aligned}$$

**Math 180, Exam 2, Study Guide**
**Problem 13 Solution**

13. Compute the following indefinite integrals:

$$\int \left(x^2 - 5x + 6\right)\,dx \qquad \int \sqrt[3]{x}\left(x^2 - \sqrt{x}\right)\,dx \qquad \int e^{3x}\,dx$$

**Solution**: Using the linearity and power rules, the first integral is:

$$\begin{aligned}\int \left(x^2 - 5x + 6\right)\,dx &= \int x^2\,dx - 5\int x\,dx + 6\int dx \\ &= \frac{1}{3}x^3 - 5\left(\frac{1}{2}x^2\right) + 6(x) + C \\ &= \boxed{\frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x + C}\end{aligned}$$

Using some algebra and the linearity and power rules, the second integral is:

$$\begin{aligned}\int \sqrt[3]{x}\left(x^2 - \sqrt{x}\right)\,dx &= \int x^{1/3}\left(x^2 - x^{1/2}\right)\,dx \\ &= \int \left(x^{7/3} - x^{5/6}\right)\,dx \\ &= \boxed{\frac{3}{10}x^{10/3} - \frac{6}{11}x^{11/6} + C}\end{aligned}$$

Using the rule $\displaystyle\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$, the third integral is:

$$\int e^{3x}\,dx = \boxed{\frac{1}{3}e^{3x} + C}$$

**Math 180, Exam 2, Study Guide**
**Problem 14 Solution**

14. Consider the function $f(x) = x^2 - x$ on $[0, 2]$. Compute $L_4$ and $R_4$.

**Solution**: For each estimate, the value of $\Delta x$ is:

$$\Delta x = \frac{b-a}{N} = \frac{2-0}{4} = \frac{1}{2}$$

The $L_4$ estimate is:

$$\begin{aligned} L_4 &= \Delta x \left[ f(0) + f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) \right] \\ &= \frac{1}{2}\left[ (0^2 - 0) + \left( \left(\frac{1}{2}\right)^2 - \frac{1}{2} \right) + (1^2 - 1) + \left( \left(\frac{3}{2}\right)^2 - \frac{3}{2} \right) \right] \\ &= \frac{1}{2}\left[ 0 - \frac{1}{4} + 0 + \frac{3}{4} \right] \\ &= \boxed{\frac{1}{4}} \end{aligned}$$

The $R_4$ estimate is:

$$\begin{aligned} R_4 &= \Delta x \left[ f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2) \right] \\ &= \frac{1}{2}\left[ \left( \left(\frac{1}{2}\right)^2 - \frac{1}{2} \right) + (1^2 - 1) + \left( \left(\frac{3}{2}\right)^2 - \frac{3}{2} \right) + (2^2 - 2) \right] \\ &= \frac{1}{2}\left[ -\frac{1}{4} + 0 + \frac{3}{4} + 2 \right] \\ &= \boxed{\frac{5}{4}} \end{aligned}$$

**Math 180, Exam 2, Study Guide**
**Problem 15 Solution**

15. Use the Fundamental Theorem of Calculus in order to compute the following integrals:

$$\int_0^2 \left(x^2 + x + 1\right)\,dx \qquad \int_1^4 \sqrt{x}\,dx \qquad \int_0^\pi \sin(2x)\,dx$$

**Solution**: The first integral has the value:

$$\begin{aligned}
\int_0^2 \left(x^2 + x + 1\right)\,dx &= \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 + x\right]_0^2 \\
&= \left[\frac{1}{3}2^3 + \frac{1}{2}2^2 + 2\right] - \left[\frac{1}{3}0^3 + \frac{1}{2}0^2 + 0\right] \\
&= \left[\frac{8}{3} + 2 + 2\right] - [0 + 0 + 0] \\
&= \boxed{\frac{20}{3}}
\end{aligned}$$

The second integral has the value:

$$\begin{aligned}
\int_1^4 \sqrt{x}\,dx &= \int_1^4 x^{1/2}\,dx \\
&= \left[\frac{2}{3}x^{3/2}\right]_1^4 \\
&= \frac{2}{3}4^{3/2} - \frac{2}{3}1^{3/2} \\
&= \frac{16}{3} - \frac{2}{3} \\
&= \boxed{\frac{14}{3}}
\end{aligned}$$

The third integral has the value:

$$\begin{aligned}
\int_0^\pi \sin(2x)\,dx &= \left[-\frac{1}{2}\cos(2x)\right]_0^\pi \\
&= \left[-\frac{1}{2}\cos(2\pi)\right] - \left[-\frac{1}{2}\cos(2(0))\right] \\
&= \left[-\frac{1}{2}\right] - \left[-\frac{1}{2}\right] \\
&= \boxed{0}
\end{aligned}$$