Math 180, Final Exam, Fall 2007
Problem 1 Solution

1. Differentiate with respect to x. Write your answers showing the use of the appropriate techniques. Do **not** simplify.

(a) $x^{2007} - x^{2/3}$ (b) $(x^2 - 2x + 2)e^x$ (c) $\ln(x^2 + 4)$

Solution:

(a) Use the Power Rule.

$$(x^{2007} - x^{2/3})' = \boxed{2007x^{2006} - \frac{2}{3}x^{-1/3}}$$

(b) Use the Product Rule

$$\begin{aligned}[(x^2 - 2x + 2)e^x]' &= (x^2 - 2x + 2)(e^x)' + e^x(x^2 - 2x + 2)' \\ &= \boxed{(x^2 - 2x + 2)e^x + e^x(2x - 2)}\end{aligned}$$

(c) Use the Chain Rule.

$$\begin{aligned}[\ln(x^2 + 4)]' &= \frac{1}{x^2 + 4} \cdot (x^2 + 4)' \\ &= \boxed{\frac{1}{x^2 + 4} \cdot 2x}\end{aligned}$$

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Problem 2 Solution

2. For the curve $y^2 + xy - x^3 = 5$

(a) use implicit differentiation to find the derivative $\dfrac{dy}{dx}$,

(b) find the equation of the line tangent to this curve at the point $(1, 2)$.

Solution:

(a) To find $\dfrac{dy}{dx}$, we use implicit differentiation.

$$\begin{aligned} y^2 + xy - x^3 &= 5 \\ \frac{d}{dx}y^2 + \frac{d}{dx}(xy) - \frac{d}{dx}x^3 &= \frac{d}{dx}5 \\ 2y\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right) - 3x^2 &= 0 \\ 2y\frac{dy}{dx} + x\frac{dy}{dx} &= 3x^2 - y \\ \frac{dy}{dx}(2y + x) &= 3x^2 - y \\ \frac{dy}{dx} &= \boxed{\frac{3x^2 - y}{2y + x}} \end{aligned}$$

(b) The value of $\dfrac{dy}{dx}$ at $(1, 2)$ is the slope of the tangent line at the point $(1, 2)$.

$$\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{3(1)^2 - 2}{2(2) + 1} = \frac{1}{5}$$

An equation for the tangent line is then:

$$\boxed{y - 2 = \frac{1}{5}(x - 1)}$$

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Problem 3 Solution

3. Use calculus to find the exact x-coordinates of any local maxima, local minima, and inflection points of the function $f(x) = 3x^5 - 20x^3 + 14$.

Solution: The critical points of $f(x)$ are the values of x for which either $f'(x)$ does not exist or $f'(x) = 0$. Since $f(x)$ is a polynomial, $f'(x)$ exists for all $x \in \mathbb{R}$ so the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(3x^5 - 20x^3 + 14\right)' &= 0 \\ 15x^4 - 60x^2 &= 0 \\ 15x^2(x^2 - 4) &= 0 \\ 15x^2(x-2)(x+2) &= 0 \\ x &= 0,\ x = \pm 2 \end{aligned}$$

Thus, $\boxed{x = 0}$ and $\boxed{x = \pm 2}$ are the critical points of f. We will use the First Derivative Test to classify the points as either local maxima or a local minima. We take the domain of $f(x)$ and split it into the intervals $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, and $(2, \infty)$ and then evaluate $f'(x)$ at a test point in each interval.

Interval	**Test Number,** c	$f'(c)$	**Sign of** $f'(c)$
$(-\infty, -2)$	-3	675	$+$
$(-2, 0)$	-1	-45	$-$
$(0, 2)$	1	-45	$-$
$(2, \infty)$	3	675	$+$

Since the sign of $f'(x)$ changes sign from $+$ to $-$ at $x = -2$, the point $f(-2) = 78$ is a local maximum and since the sign of $f'(x)$ changes from $-$ to $+$ at $x = 2$, the point $f(2) = -50$ is a local minimum. The sign of $f'(x)$ does not change at $x = 0$ so $f(0) = 0$ is neither a local maximum nor a local minimum.

The critical points of $f(x)$ are the values of x where $f''(x)$ changes sign. To determine these we first find the values of x for which $f''(x) = 0$.

$$\begin{aligned} f''(x) &= 0 \\ \left(15x^4 - 60x^2\right)' &= 0 \\ 60x^3 - 120x &= 0 \\ 60x(x^2 - 2) &= 0 \\ x &= 0,\ x = \pm\sqrt{2} \end{aligned}$$

We now take the domain of $f(x)$ and split it into the intervals $(-\infty, -\sqrt{2})$, $(-\sqrt{2}, 0)$, $(0, \sqrt{2})$, and $(\sqrt{2}, \infty)$ and then evaluate $f''(x)$ at a test point in each interval.

Interval	**Test Number,** c	$f''(c)$	**Sign of** $f''(c)$
$(-\infty, -\sqrt{2})$	-2	-240	$-$
$(-\sqrt{2}, 0)$	-1	60	$+$
$(0, \sqrt{2})$	1	-60	$-$
$(\sqrt{2}, \infty)$	2	240	$+$

We see that $f''(x)$ changes sign at $x = 0$ and $x = \pm\sqrt{2}$. Thus, these are inflection points.

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Problem 4 Solution

4. Find
$$\lim_{x\to 0} \frac{e^x - x - 1}{3x^2}$$
Explain how you obtain your answer.

Solution: Upon substituting $x = 0$ into the function we find that

$$\frac{e^x - x - 1}{3x^2} = \frac{e^0 - 0 - 1}{3(0)^2} = \frac{0}{0}$$

which is indeterminate. We resolve this indeterminacy by using L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - x - 1}{3x^2} &\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{(e^x - x - 1)'}{(3x^2)'} \\
&= \lim_{x\to 0} \frac{e^x - 1}{6x}
\end{aligned}$$

Substituting $x = 0$ gives us the indeterminate form $\frac{0}{0}$ again. Thus, we apply L'Hôpital's Rule one more time.

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - 1}{6x} &\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{(e^x - 1)'}{(6x)'} \\
&= \lim_{x\to 0} \frac{e^x}{6} \\
&= \frac{e^0}{6} \\
&= \boxed{\frac{1}{6}}
\end{aligned}$$

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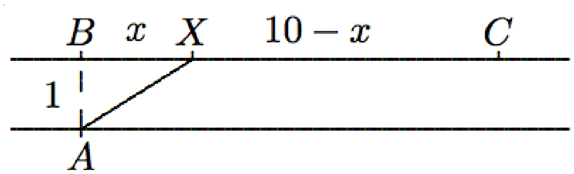
Problem 5 Solution

5. An electrical company at point A needs to run a wire from a generator to a factory that is on the other side of a one mile wide river and 10 miles downstream at a point C. It costs \$600 per mile to run the wire on towers across the river and \$400 per mile to run the wire over land along the river. The wire will cross the river from A to a point X and then travel over land from X to C. Let x be the distance from B to X.

(a) Find the total cost as a function $f(x)$ of the variable x.

(b) Use calculus to find the value of x that minimizes the cost.



Solution:

(a) The cost of the wire is:

$$\begin{aligned}
\text{Cost} &= A \text{ to } X + X \text{ to } C \\
\text{Cost} &= \$600(\text{distance from } A \text{ to } X) + \$400(\text{distance from } X \text{ to } C) \\
f(x) &= 600\sqrt{x^2+1} + 400(10-x)
\end{aligned}$$

The domain of $f(x)$ is $[0, 10]$.

(b) The cost is a minimum either at a critical point of $f(x)$ or at one of the endpoints of $[0, 10]$. The critical points are solutions to $f'(x) = 0$.

$$\begin{aligned}
f'(x) &= 0 \\
\left(600\sqrt{x^2+1} + 400(10-x)\right)' &= 0 \\
600 \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot (x^2+1)' + 400(10-x)' &= 0 \\
\frac{300}{\sqrt{x^2+1}} \cdot (2x) + 400(-1) &= 0 \\
\frac{3x}{\sqrt{x^2+1}} - 2 &= 0 \\
2\sqrt{x^2+1} &= 3x \\
(2\sqrt{x^2+1})^2 &= (3x)^2 \\
4(x^2+1) &= 9x^2 \\
x^2 &= \frac{4}{5} \\
x &= \frac{2}{\sqrt{5}}
\end{aligned}$$

The cost at $x = \frac{2}{\sqrt{5}}$ is:

$$f\left(\frac{2}{\sqrt{5}}\right) = 200(20 + \sqrt{5})$$

The costs at the endpoints are:

$$f(0) = 4600$$
$$f(10) = 600\sqrt{101}$$

both of which are larger than $200(20 + \sqrt{5})$. Thus, the cost is an absolute minimum on $[0, 10]$ when $x = \frac{2}{\sqrt{5}}$.

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Problem 6 Solution

6. Differentiate with respect to x. Write your answers showing the use of the appropriate techniques. Do **not** simplify.

(a) $\dfrac{x^2+1}{x^2+x+1}$ (b) $\sin^3(5x+2)$ (c) $\arctan\left(\dfrac{x}{2}\right)$

Solution:

(a) Use the Quotient Rule.

$$\begin{aligned}\left(\frac{x^2+1}{x^2+x+1}\right)' &= \frac{(x^2+x+1)(x^2+1)' - (x^2+1)(x^2+x+1)'}{(x^2+x+1)^2} \\ &= \boxed{\frac{(x^2+x+1)(2x) - (x^2+1)(2x+1)}{(x^2+x+1)^2}}\end{aligned}$$

(b) Use the Chain Rule.

$$\begin{aligned}[\sin^3(5x+2)]' &= 3\sin^2(5x+2)\cdot(5x+2)' \\ &= \boxed{3\sin^2(5x+2)\cdot 5}\end{aligned}$$

(c) Use the Chain Rule.

$$\begin{aligned}\left[\arctan\left(\frac{x}{2}\right)\right]' &= \frac{1}{1+(\frac{x}{2})^2}\cdot\left(\frac{x}{2}\right)' \\ &= \boxed{\frac{1}{1+\frac{x^2}{4}}\cdot\frac{1}{2}}\end{aligned}$$

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Problem 7 Solution

7.

(a) Calculate the left and right Riemann sums with three subdivisions, L_3 and R_3, for the integral:

$$\int_0^6 f(x)\,dx$$

Some values of the function f are given in the table:

x	0	2	4	6
$f(x)$	1.6	1.9	2.4	3.1

(b) If the function f is increasing, could the integral be greater than 15? Explain why or why not.

Solution:

(a) In calculating L_3 and R_3, the value of Δx is:

$$\Delta x = \frac{b-a}{N} = \frac{6-0}{3} = 2$$

The integral estimates are then:

$$\begin{aligned} L_3 &= \Delta x\,[f(0)+f(2)+f(4)] \\ &= 2\,[1.6+1.9+2.4] \\ &= \boxed{11.8} \\ R_3 &= \Delta x\,[f(2)+f(4)+f(6)] \\ &= 2\,[1.9+2.4+3.1] \\ &= \boxed{14.8} \end{aligned}$$

(b) Since f is increasing, we know that $L_3 \leq S \leq R_3$ where S is the actual value of the integral. Since $R_3 = 14.8$, it is not possible that $S > 15$.

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Problem 8 Solution

8.

(a) Write the integral which gives the area of the region between $x = 0$ and $x = 1$, above the x-axis, and below the curve $y = x - x^3$.

(b) Evaluate your integral exactly to find the area.

Solution:

(a) The area of the region is given by the integral:

$$\int_0^1 (x - x^3)\,dx$$

(b) We use FTC I to evaluate the integral.

$$\begin{aligned}\int_0^1 (x - x^3)\,dx &= \frac{x^2}{2} - \frac{x^4}{4}\bigg|_0^1 \\ &= \left(\frac{1^2}{2} - \frac{1^4}{4}\right) - \left(\frac{0^2}{2} - \frac{0^4}{4}\right) \\ &= \boxed{\frac{1}{4}}\end{aligned}$$

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Problem 9 Solution

9. Evaluate the integral $\int_0^1 \frac{1}{\sqrt{3x+1}}\,dx$ by finding an antiderivative.

Solution: We evaluate the integral using the substitution $u = 3x+1$, $\frac{1}{3}\,du = dx$. The limits of integration become $u = 3(0) + 1 = 1$ and $u = 3(1) + 1 = 4$. Making these substitutions and evaluating the integral we get:

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{3x+1}}\,dx &= \frac{1}{3}\int_1^4 \frac{1}{\sqrt{u}}\,du \\
&= \frac{1}{3}\Big[2\sqrt{u}\,\Big]_1^4 \\
&= \frac{1}{3}\Big[2\sqrt{4} - 2\sqrt{1}\Big] \\
&= \boxed{\frac{2}{3}}
\end{aligned}$$

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Problem 10 Solution

10. The graph below represents the derivative, $f'(x)$.

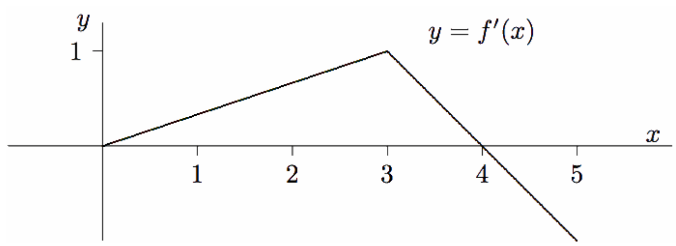
(a) On what interval is f increasing?

(b) On what interval is f decreasing?

(c) For what value of x is $f(x)$ a maximum?

(d) What is $\int_0^5 f'(x)\,dx$?

(e) What is $f(5) - f(0)$?



Solution:

(a) f is increasing on $(0, 4)$ because $f'(x) > 0$ on this interval.

(b) f is decreasing on $(4, 5)$ because $f'(x) < 0$ on this interval.

(c) f has a local maximum at $x = 4$ because $f'(4) = 0$ and f' changes sign from $+$ to $-$ at $x = 4$.

(d) The value of $\int_0^5 f'(x)\,dx$ represents the signed area between $y = f(x)$ and the x-axis on the interval $[0, 5]$. Using geometry, we find that:

$$\int_0^5 f'(x)\,dx = \frac{1}{2}(4)(1) - \frac{1}{2}(1)(1) = \boxed{\frac{3}{2}}$$

assuming that the value of $f'(5)$ is -1.

(e) The value of $f(5) - f(0)$ is found using the fact that an antiderivative of $f'(x)$ is $f(x)$. That is, $\int f'(x)\,dx = f(x)$. Thus, from the Fundamental Theorem of Calculus, Part I we have:

$$\int_0^5 f'(x)\,dx = f(x)\Big|_0^5 = f(5) - f(0) = \boxed{\frac{3}{2}}$$