Math 180, Final Exam, Fall 2008
Problem 1 Solution

1. Differentiate with respect to x. Write your answers showing the use of the appropriate techniques. Do **not** simplify.

(a) $e^x \sin(x)$ (b) $\ln(\sqrt{x}+8)$ (c) $\dfrac{x^3-1}{x^2+1}$

Solution:

(a) Use the Product Rule.

$$\begin{aligned}(e^x \sin(x))' &= e^x(\sin(x))' + (e^x)'\sin(x) \\ &= \boxed{e^x \cos x + e^x \sin x}\end{aligned}$$

(b) Use the Chain Rule.

$$\begin{aligned}\left[\ln(\sqrt{x}+8)\right]' &= \frac{1}{\sqrt{x}+8}\cdot(\sqrt{x}+8)' \\ &= \boxed{\frac{1}{\sqrt{x}+8}\cdot\left(\frac{1}{2\sqrt{x}}\right)}\end{aligned}$$

(c) Use the Quotient Rule.

$$\begin{aligned}\left(\frac{x^3-1}{x^2+1}\right)' &= \frac{(x^2+1)(x^3-1)' - (x^3-1)(x^2+1)'}{(x^2+1)^2} \\ &= \boxed{\frac{(x^2+1)(3x^2) - (x^3-1)(2x)}{(x^2+1)^2}}\end{aligned}$$

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Problem 2 Solution

2. Let $f(x) = x + x^3$.

(a) Find $f(1)$, $f'(1)$, and $f''(1)$.

(b) Find the equation of the line tangent to the graph of f at $x = 1$.

(c) Is f concave up or down at $x = 1$?

Solution:

(a) The first two derivatives of f are found using the Power Rule.

$$f'(x) = 1 + 3x^2, \qquad f''(x) = 6x$$

The values of f, f', and f'' at $x = 1$ are:

$$\boxed{f(1) = 2, \quad f'(1) = 4, \quad f''(1) = 6}$$

(b) The equation of the line tangent to f at $x = 1$ is:

$$\boxed{y - 2 = 4(x - 1)}$$

(c) Since $f''(1) = 6 > 0$, we know that f is concave up at $x = 1$.

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Problem 3 Solution

3. For the curve $x^2 + xy + y^3 = 1$, use implicit differentiation to find the derivative $\frac{dy}{dx}$ when $x = -1$, $y = 1$.

Solution: We must find $\frac{dy}{dx}$ using implicit differentiation.

$$\begin{aligned} x^2 + xy + y^3 &= 1 \\ \frac{d}{dx}x^2 + \frac{d}{dx}(xy) + \frac{d}{dx}y^3 &= \frac{d}{dx}1 \\ 2x + \left(x\frac{dy}{dx} + y\right) + 3y^2\frac{dy}{dx} &= 0 \\ x\frac{dy}{dx} + 3y^2\frac{dy}{dx} &= -2x - y \\ \frac{dy}{dx}\left(x + 3y^2\right) &= -2x - y \\ \frac{dy}{dx} &= \frac{-2x - y}{x + 3y^2} \end{aligned}$$

The value of $\frac{dy}{dx}$ at $(-1, 1)$ is:

$$\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{-2(-1) - 1}{-1 + 3(1)^2} = \boxed{\frac{1}{2}}$$

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Problem 4 Solution

4. Find an antiderivative for $f(x) = \frac{1}{\sqrt{x}} + \sqrt{x}$, that is, find $\int \left(\frac{1}{\sqrt{x}} + \sqrt{x} \right) \, dx$.

Solution: An antiderivative for $f(x)$ is:

$$\begin{aligned}
\int \left(\frac{1}{\sqrt{x}} + \sqrt{x} \right) \, dx &= \int \left(x^{-1/2} + x^{1/2} \right) \, dx \\
&= \frac{x^{-1/2+1}}{-1/2+1} + \frac{x^{1/2+1}}{1/2+1} + C \\
&= \boxed{2x^{1/2} + \frac{2}{3}x^{3/2} + C}
\end{aligned}$$

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Problem 5 Solution

5. For the function $f(x) = \dfrac{x+1}{x^2+3}$,

(a) use calculus to find the exact x-coordinates of any local maxima and local minima of the function

(b) find the exact values of $f(x)$ at these points.

Solution: The critical points of $f(x)$ are the values of x for which either $f'(x)$ does not exist or $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(\frac{x+1}{x^2+3}\right)' &= 0 \\ \frac{(x^2+3)(x+1)' - (x+1)(x^2+3)'}{(x^2+3)^2} &= 0 \\ \frac{(x^2+3)(1) - (x+1)(2x)}{(x^2+3)^2} &= 0 \\ \frac{3 - 2x - x^2}{(x^2+3)^2} &= 0 \\ 3 - 2x - x^2 &= 0 \\ (3+x)(1-x) &= 0 \\ x &= 1,\ -3 \end{aligned}$$

Thus, $\boxed{\boxed{x = 1,\ -3}}$ are the critical points of f. (Note: $x^2 + 3 > 0$ for all x.)

We will use the First Derivative Test to classify the critical points. The domain of f is $(-\infty, \infty)$. We now split the domain into the intervals $(-\infty, -3)$, $(-3, 1)$, and $(1, \infty)$. We then evaluate $f'(x)$ at a test point in each interval.

Interval	**Test Point**, c	$f'(c)$	**Sign of** $f'(c)$
$(-\infty, -3)$	-4	$f'(-4) = -\frac{5}{361}$	$-$
$(-3, 1)$	0	$f'(0) = \frac{1}{3}$	$+$
$(1, \infty)$	2	$f'(2) = -\frac{5}{49}$	$-$

Since f' changes sign from $-$ to $+$ at $x = -3$ the First Derivative Test implies that $f(-3) = -\dfrac{1}{6}$ is a local minimum and since f' changes sign from $+$ to $-$ at $x = 1$ the First Derivative Test implies that $f(1) = \dfrac{1}{2}$ is a local maximum.

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Problem 6 Solution

6. Find

$$\lim_{x\to 0} \frac{e^{x^2} - \cos x}{x^2}$$

Explain how you obtain your answer.

Solution: Upon substituting $x = 0$ into the function we find that

$$\frac{e^{x^2} - \cos x}{x^2} = \frac{e^0 - \cos 0}{0^2} = \frac{0}{0}$$

which is indeterminate. We resolve this indeterminacy by using L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to 0} \frac{e^{x^2} - \cos x}{x^2} &\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{(e^{x^2} - \cos x)'}{(x^2)'} \\
&= \lim_{x\to 0} \frac{2xe^{x^2} + \sin x}{2x} \\
&= \lim_{x\to 0} \left(\frac{2xe^{x^2}}{2x} + \frac{\sin x}{2x} \right) \\
&= \lim_{x\to 0} \left(e^{x^2} + \frac{1}{2} \cdot \frac{\sin x}{x} \right) \\
&= e^{0^2} + \frac{1}{2} \cdot 1 \\
&= \boxed{\frac{3}{2}}
\end{aligned}$$

Note: We used the fact that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ to evaluate the limit.

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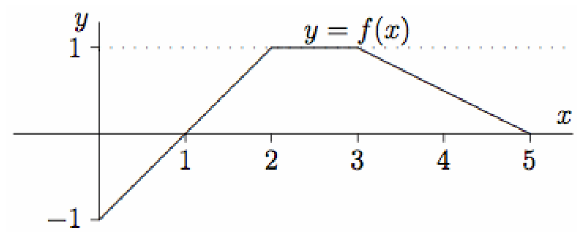
Problem 7 Solution

7. The graph of $y = f(x)$ is below.

(a) Find $\displaystyle\int_0^5 f(x)\,dx$.

(b) If $F(x) = \displaystyle\int_0^x f(t)\,dt$, find $F'(3)$.



Solution:

(a) The value of $\displaystyle\int_0^5 f(x)\,dx$ is the signed area between the graph of $y = f(x)$ and the x-axis on the interval $[0,5]$. We use the additivity of integrals to break the integral down as follows:

$$\int_0^5 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^5 f(x)\,dx$$

The reason for this is that the regions between $y = f(x)$ and the x-axis on the intervals $[0,1]$, $[1,2]$, $[2,3]$, and $[3,5]$ are either triangles or rectangles. The signed area is then:

$$\begin{aligned}\int_0^5 f(x)\,dx &= \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^5 f(x)\,dx \\ &= -\frac{1}{2}(1)(1) + \frac{1}{2}(1)(1) + (1)(1) + \frac{1}{2}(2)(1) \\ &= \boxed{2}\end{aligned}$$

(b) Using the Fundamental Theorem of Calculus, Part II we know that $F'(x) = f(x)$. Then the value of $F'(3)$ is $f(3)$ which is the y-coordinate of the point on the graph when $x = 3$. From the graph we see that $F'(3) = f(3) = \boxed{1}$.

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Problem 8 Solution

8.

(a) Write the integral which gives the area of the region between $x = 0$ and $x = \pi$, above the x-axis, and below the curve $y = \sin(x)$.

(b) Evaluate your integral exactly to find the area.

Solution:

(a) The area of the region is given by the integral:

$$\int_0^\pi \sin(x)\,dx$$

(b) We use FTC I to evaluate the integral.

$$\begin{aligned}\int_0^\pi \sin(x)\,dx &= -\cos(x)\Big|_0^\pi \\ &= -\cos\pi - (-\cos 0) \\ &= -(-1) - (-1) \\ &= \boxed{2}\end{aligned}$$

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Problem 9 Solution

9. Evaluate the integral $\int xe^{x^2}\,dx$.

Solution: We use the substitution $u = x^2$, $\frac{1}{2}\,du = x\,dx$. Making the substitutions and evaluating the integral we get:

$$\begin{aligned}\int xe^{x^2}\,dx &= \frac{1}{2}\int e^u\,du \\ &= \frac{1}{2}e^u + C \\ &= \boxed{\frac{1}{2}e^{x^2} + C}\end{aligned}$$

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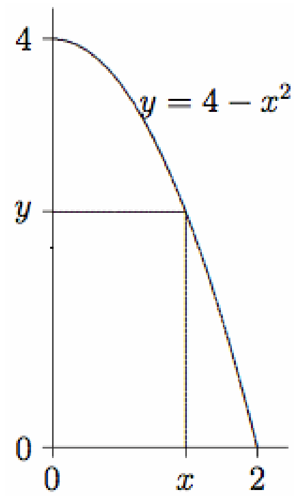
Problem 10 Solution

10. Find the dimensions and area of the rectangle of maximum area with corners at $(0,0)$, $(x,0)$, and (x,y) where $y = 4 - x^2$. (The maximum will occur for a value of x with $0 < x < 2$.)



Solution: The dimensions of the rectangle are x and y. Therefore, the area of the rectangle has the equation:

$$\text{Area} = xy \tag{1}$$

We must find an equation that relates x to y so that we can eliminate y from the area equation. This equation is

$$y = 4 - x^2 \tag{2}$$

because (x, y) must lie on this line. Plugging this into the area equation we get:

$$\begin{aligned} \text{Area} &= x\left(4 - x^2\right) \\ f(x) &= 4x - x^3 \end{aligned}$$

We seek the value of x that maximizes $f(x)$. The interval in the problem is $[0, 2]$ because the upper corner of the rectangle must lie in the first quadrant.

The absolute maximum of $f(x)$ will occur either at a critical point of $f(x)$ in $[0, 2]$ or at one of the endpoints. The critical points of $f(x)$ are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(4x - x^3\right)' &= 0 \\ 4 - 3x^2 &= 0 \\ x^2 &= \frac{4}{3} \\ \boxed{x = \frac{2}{\sqrt{3}}} \end{aligned}$$

Plugging this into $f(x)$ we get:

$$f\left(\frac{2}{\sqrt{3}}\right) = 4\left(\frac{2}{\sqrt{3}}\right) - \left(\frac{2}{\sqrt{3}}\right)^3 = \frac{16}{3\sqrt{3}}$$

Evaluating $f(x)$ at the endpoints $x = 0$ and $x = 2$ we get:

$$f(0) = 4(0) - 0^3 = 0$$
$$f(2) = 4(2) - 2^3 = 0$$

both of which are smaller than $\frac{16}{3\sqrt{3}}$. We conclude that the area is an absolute maximum at $x = \frac{2}{\sqrt{3}}$ and that the resulting area is $\frac{16}{3\sqrt{3}}$. The last step is to find the corresponding value for y by plugging $x = 3$ into equation (2).

$$y = 4 - \left(\frac{2}{\sqrt{3}}\right)^2 = \boxed{\frac{8}{3}}$$