Math 180, Final Exam, Fall 2010
Problem 1 Solution

1. Differentiate each function with respect to x. Leave your answer in unsimplified form.

(a) $\dfrac{e^x - x^3 - 1}{2x}$

(b) $\ln(\cos(x))$

(c) $3^x + x^3$

Solution:

(a) Use the Quotient Rule.

$$\left(\frac{e^x - x^3 - 1}{2x}\right)' = \frac{(2x)(e^x - x^3 - 1)' - (e^x - x^3 - 1)(2x)'}{(2x)^2}$$
$$= \boxed{\frac{(2x)(e^x - 3x^2) - (e^x - x^3 - 1)(2)}{(2x)^2}}$$

(b) Use the Chain Rule.

$$[\ln(\cos(x))]' = \frac{1}{\cos(x)} \cdot (\cos(x))'$$
$$= \boxed{\frac{1}{\cos(x)} \cdot (-\sin(x))}$$

(c) Use the Exponential and Power Rules.

$$\left(3^x + x^3\right)' = \boxed{(\ln 3)3^x + 3x^2}$$

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Problem 2 Solution

2. Suppose y is a function of x defined implicitly by the equation

$$x + 16 = y^2 x$$

(a) Use implicit differentiation to calculate the derivative $\dfrac{dy}{dx}$.

(b) Find the equation of the tangent line to this curve at the point $(2, 3)$.

Solution:

(a) We find $\dfrac{dy}{dx}$ using implicit differentiation.

$$\begin{aligned}
x + 16 &= y^2 x \\
\frac{d}{dx}(x) + \frac{d}{dx}(16) &= \frac{d}{dx}\left(y^2 x\right) \\
1 + 0 &= y^2 \frac{d}{dx}(x) + x\frac{d}{dx}\left(y^2\right) \\
1 &= y^2(1) + x\left(2y\frac{dy}{dx}\right) \\
1 &= y^2 + 2xy\frac{dy}{dx} \\
2xy\frac{dy}{dx} &= 1 - y^2 \\
\frac{dy}{dx} &= \frac{1 - y^2}{2xy}
\end{aligned}$$

(b) The value of $\dfrac{dy}{dx}$ at $(2, 3)$ is the slope of the tangent line.

$$\left.\frac{dy}{dx}\right|_{(2,3)} = \frac{1 - 3^2}{2(2)(3)} = -\frac{2}{3}$$

An equation for the tangent line at $(2, 3)$ is then:

$$\boxed{y - 3 = -\frac{2}{3}(x - 2)}$$

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Problem 3 Solution

3. Calculate each limit and indicate the method used:

(a) $\lim_{x \to 1} \frac{x^2 - 5x + 4}{1 - x}$

(b) $\lim_{x \to 0} \frac{x \cos(x)}{\sin(x)}$

Solution:

(a) Upon substituting $x = 1$ into the function $f(x) = \frac{x^2 - 5x + 4}{1 - x}$ we find that

$$\frac{x^2 - 5x + 4}{1 - x} = \frac{1^2 - 5(1) + 4}{1 - 1} = \frac{0}{0}$$

which is indeterminate. We can resolve the indeterminacy by factoring the numerator of $f(x)$.

$$\lim_{x \to 1} \frac{x^2 - 5x + 4}{1 - x} = \lim_{x \to 1} \frac{(1 - x)(4 - x)}{1 - x} = \lim_{x \to 1} (4 - x) = 4 - 1 = \boxed{3}$$

In the final step above we were able to plug in $x = 1$ by using the fact that the function $4 - x$ is continuous at $x = 1$.

(b) Upon substituting $x = 0$ into the function $f(x) = \frac{x \cos(x)}{\sin(x)}$ we find that

$$\frac{x \cos(x)}{\sin(x)} = \frac{0 \cdot \cos(0)}{\sin(0)} = \frac{0}{0}$$

which is indeterminate. We can resolve the indeterminacy by rewriting the limit. Using the multiplication rule for limits:

$$\lim_{x \to c} f(x)g(x) = \left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right)$$

we get:

$$\begin{aligned}
\lim_{x \to 0} \frac{x \cos(x)}{\sin(x)} &= \lim_{x \to 0} \cos(x) \cdot \frac{x}{\sin(x)} \\
&= \left(\lim_{x \to 0} \cos(x)\right)\left(\lim_{x \to 0} \frac{x}{\sin(x)}\right) \\
&= \left(\lim_{x \to 0} \cos(x)\right)\left(\lim_{x \to 0} \frac{1}{\frac{\sin x}{x}}\right)
\end{aligned}$$

We now use the quotient rule for limits:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$$

to get:

$$\begin{aligned}
\lim_{x \to 0} \frac{x \cos(x)}{\sin(x)} &= \left(\lim_{x \to 0} \cos(x)\right) \left(\lim_{x \to 0} \frac{1}{\frac{\sin x}{x}}\right) \\
&= \left(\lim_{x \to 0} \cos(x)\right) \left(\frac{\lim_{x \to 0} 1}{\lim_{x \to 0} \frac{\sin x}{x}}\right)
\end{aligned}$$

Finally, we use the fact that:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

to get our final answer:

$$\begin{aligned}
\lim_{x \to 0} \frac{x \cos(x)}{\sin(x)} &= \left(\lim_{x \to 0} \cos(x)\right) \left(\frac{\lim_{x \to 0} 1}{\lim_{x \to 0} \frac{\sin x}{x}}\right) \\
&= (\cos(0)) \cdot \left(\frac{1}{1}\right) \\
&= \boxed{1}
\end{aligned}$$

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Problem 4 Solution

4.

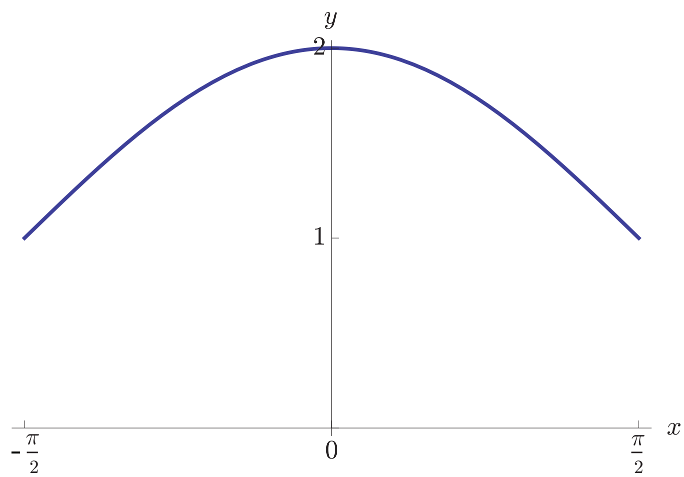
(a) Sketch the curve $y = 1 + \cos(x)$ for $-\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$.

(b) Write an integral that gives the area under the curve $y = 1 + \cos(x)$, above the x-axis, and between $x = -\dfrac{\pi}{2}$ and $x = \dfrac{\pi}{2}$.

(c) Compute the integral from part (b).

Solution:

(a) A sketch of the curve is shown below.



(b) An integral that gives the area under the curve is:

$$\text{Area} = \int_{-\pi/2}^{\pi/2} (1 + \cos(x))\; dx$$

(c) The value of the integral is obtained as follows:

$$\begin{aligned}
\text{Area} &= \int_{-\pi/2}^{\pi/2} (1 + \cos(x))\; dx \\
&= \Big[x + \sin(x)\Big]_{-\pi/2}^{\pi/2} \\
&= \left[\frac{\pi}{2} + \sin\left(\frac{\pi}{2}\right)\right] - \left[-\frac{\pi}{2} + \sin\left(-\frac{\pi}{2}\right)\right] \\
&= \left[\frac{\pi}{2} + 1\right] - \left[-\frac{\pi}{2} - 1\right] \\
&= \boxed{\pi + 2}
\end{aligned}$$

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Problem 5 Solution

5. Define $f(x) = \begin{cases} -x^2 - 2x & \text{if } x \leq 0 \\ x^3 - 2x & \text{if } x > 0 \end{cases}$.

(a) Is $f(x)$ continuous at $x = 0$? Is it differentiable at $x = 0$? Justify your answer.

(b) Locate all critical points of $f(x)$.

(c) Find the maximum and minimum values of $f(x)$ on the interval $[-2, 2]$.

Solution:

(a) We start by computing the one-sided limits $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left(x^3 - 2x\right) = 0^3 - 2(0) = 0$$
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \left(-x^2 - 2x\right) = -0^2 - 2(0) = 0$$

The one-sided limits are the same and both are equal to 0. Thus, $\lim_{x\to 0} f(x) = 0$. Since $f(x)$ is defined as $-x^2 - 2x$ when $x = 0$ we have:

$$f(0) = -0^2 - 2(0) = 0$$

Therefore, since $\lim_{x\to 0} f(x) = f(0) = 0$ we know that $f(x)$ is continuous at $x = 0$.

By definition, the derivative of $f(x)$ at $x = 0$ is:

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

In order for $f'(0)$ to exist, the limit on the right hand side must exist which means that the one-sided limits must be the same. The one-sided limits are:

$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \frac{d}{dx}\left(x^3 - 2x\right)\Big|_{x=0} = \left(3x^2 - 2\right)\Big|_{x=0} = -2$$

and

$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \frac{d}{dx}\left(-x^2 - 2x\right)\Big|_{x=0} = \left(-2x - 2\right)\Big|_{x=0} = -2$$

Since the one-sided limits are the same and both are equal to -2, we know that $f(x)$ is differentiable at $x = 0$ and $f'(0) = -2$.

(b) The critical points of $f(x)$ will occur at values of x where either $f'(x) = 0$ or $f'(x)$ does not exist. We showed in part (a) that $f'(0)$ exists. $f'(x)$ exists for all other values of x. Therefore, the critical points are values of x for which $f'(x) = 0$.

On the interval $x \leq 0$ we have:

$$\begin{aligned} f'(x) &= 0 \\ \frac{d}{dx}\left(-x^2 - 2x\right) &= 0 \\ -2x - 2 &= 0 \\ x &= -1 \end{aligned}$$

On the interval $x > 0$ we have:

$$\begin{aligned} f'(x) &= 0 \\ \frac{d}{dx}\left(x^3 - 2x\right) &= 0 \\ 3x^2 - 2 &= 0 \\ x^2 &= \frac{2}{3} \\ x &= \sqrt{\frac{2}{3}} \end{aligned}$$

(c) To find the minimum and maximum values of $f(x)$ on the interval $[-2, 2]$, we evaluate $f(x)$ at $x = -2, -1, \sqrt{\frac{2}{3}}, 2$.

$$\begin{aligned} f(-2) &= -(-2)^2 - 2(-2) = 0 \\ f(-1) &= -(-1)^2 - 2(-1) = 1 \\ f(\sqrt{\tfrac{2}{3}}) &= (-\sqrt{\tfrac{2}{3}})^3 - 2(\sqrt{\tfrac{2}{3}}) = -\tfrac{4}{3}(\sqrt{\tfrac{2}{3}}) \\ f(2) &= 2^3 - 2(2) = 4 \end{aligned}$$

The largest value is 4 and the smallest value is $-\frac{4}{3}(\sqrt{\frac{2}{3}})$. Thus, these are the maximum and minimum values of $f(x)$, respectively.

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Problem 6 Solution

6. Calculate the antiderivatives:

(a) $\displaystyle\int \left(\frac{e^x + e^{-x}}{2}\right) dx$

(b) $\displaystyle\int \left(\frac{\cos(\frac{1}{x})}{x^2}\right) dx$

Solution:

(a) We begin by splitting the integral into two integrals and pulling out the constant.

$$\begin{aligned}\int \left(\frac{e^x + e^{-x}}{2}\right) dx &= \int \frac{e^x}{2}\, dx + \int \frac{e^{-x}}{2}\, dx \\ &= \frac{1}{2}\int e^x\, dx + \frac{1}{2}\int e^{-x}\, dx\end{aligned}$$

Then we use the exponential rule $\int e^{kx}\, dx = \frac{1}{k}e^{kx} + C$ to evaluate the integrals on the right hand side.

$$\begin{aligned}\int \left(\frac{e^x + e^{-x}}{2}\right) dx &= \frac{1}{2}\int e^x\, dx + \frac{1}{2}\int e^{-x}\, dx \\ &= \boxed{\frac{1}{2}e^x - \frac{1}{2}e^{-x} + C}\end{aligned}$$

(b) We use the substitution $u = \frac{1}{x}$, $-\,du = \frac{1}{x^2}\, dx$. Making the substitutions and evaluating the integral we get:

$$\begin{aligned}\int \left(\frac{\cos(\frac{1}{x})}{x^2}\right) dx &= \int \cos\left(\frac{1}{x}\right) \cdot \frac{1}{x^2}\, dx \\ &= \int \cos(u) \cdot (-\,du) \\ &= -\int \cos(u)\, du \\ &= -\sin(u) + C \\ &= \boxed{-\sin\left(\frac{1}{x}\right) + C}\end{aligned}$$

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Problem 7 Solution

7. Let $g(x) = xe^{-x}$.

(a) Find the critical points of g and determine the intervals where g is increasing and decreasing.

(b) Find the inflection points of g and determine the intervals where g is concave up and concave down.

Solution:

(a) We begin by finding the critical points of $g(x)$. The critical points of $g(x)$ are the values of x for which either $g'(x)$ does not exist or $g'(x) = 0$. Since $g(x)$ is the quotient of a polynomial and e^x, we know that $g'(x)$ exists for all $x \in \mathbb{R}$ so the only critical points are solutions to $g'(x) = 0$.

$$\begin{aligned} g'(x) &= 0 \\ \left(xe^{-x}\right)' &= 0 \\ x\left(e^{-x}\right)' + e^{-x}\,(x)' &= 0 \\ -xe^{-x} + e^{-x} &= 0 \\ e^{-x}(1-x) &= 0 \\ 1 - x &= 0 \\ x &= 1 \end{aligned}$$

The domain of g is $(-\infty, \infty)$. We now split the domain into the two intervals $(-\infty, 1)$ and $(1, \infty)$. We then evaluate $g'(x)$ at a test point in each interval to determine the intervals of monotonicity.

Interval	**Test Point**, c	$g'(c)$	**Sign of** $g'(c)$
$(-\infty, 1)$	0	$g'(0) = 1$	$+$
$(1, \infty)$	2	$g'(2) = -e^{-2}$	$-$

Using the table we conclude that g is increasing on $\boxed{(-\infty, 1)}$ because $f'(x) > 0$ for all $x \in (-\infty, 1)$ and that g is decreasing on $\boxed{(1, \infty)}$ because $f'(x) < 0$ for all $x \in (1, \infty)$.

(b) To determine the intervals of concavity we start by finding solutions to the equation $g''(x) = 0$ and where $g''(x)$ does not exist. However, since $g(x)$ is the quotient of a polynomial and e^x we know that $g''(x)$ will exist for all $x \in \mathbb{R}$. The solutions to $g''(x) = 0$ are:

$$\begin{aligned} g''(x) &= 0 \\ \left(e^{-x}(1-x)\right)' &= 0 \\ e^{-x}(1-x)' + (1-x)\left(e^{-x}\right)' &= 0 \\ -e^{-x} + (1-x)\left(-e^{-x}\right) &= 0 \\ e^{-x}(-1-1+x) &= 0 \\ e^{-x}(-2+x) &= 0 \\ -2+x &= 0 \\ x &= 2 \end{aligned}$$

We now split the domain into the two intervals $(-\infty, 2)$ and $(2, \infty)$. We then evaluate $g''(x)$ at a test point in each interval to determine the intervals of concavity.

Interval	**Test Point**, c	$g''(c)$	**Sign of** $g''(c)$
$(-\infty, 2)$	1	$g''(1) = -e^{-1}$	$-$
$(2, \infty)$	3	$g''(3) = e^{-3}$	$+$

Using the table we conclude that g is concave up on $\boxed{(2, \infty)}$ because $g''(x) > 0$ for all $x \in (2, \infty)$ and that g is concave down on $\boxed{(-\infty, 2)}$ because $g''(x) < 0$ for all $x \in (-\infty, 2)$.

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Problem 8 Solution

8. Use linear approximation to estimate the value of $(63)^{1/3}$. In your solution, clearly indicate the function whose linear approximation you are using, the point where the approximation is taken, and the linear function that you evaluate at $x = 63$.

Solution: We will use the linearization formula:

$$L(x) = f(a) + f'(a)(x - a)$$

where we define $f(x) = x^{1/3}$ and $a = 64$. The derivative of $f(x)$ is $f'(x) = \frac{1}{3}x^{-2/3}$. Evaluating $f(x)$ and $f'(x)$ at $x = 64$ we get:

$$f(64) = (64)^{1/3} = 4$$
$$f'(64) = \frac{1}{3}(64)^{-2/3} = \frac{1}{48}$$

The linearization of $f(x)$ is then:

$$L(x) = 4 + \frac{1}{48}(x - 64)$$

The estimated value of $(63)^{1/3}$ is $L(63)$.

$$f(63) \approx L(63)$$
$$f(63) \approx 4 + \frac{1}{48}(63 - 64)$$
$$f(63) \approx \boxed{\frac{191}{48}}$$

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Problem 9 Solution

9. Show that there is a positive real solution of the equation $x^2 + 2 = 10^x$.

Solution: Let $f(x) = x^2 + 2 - 10^x$. First we recognize that $f(x)$ is continuous everywhere. Next, we must find an interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs. Let's choose $a = 0$ and $b = 1$.

$$f(0) = 0^2 + 2 - 10^0 = 1$$
$$f(1) = 1^2 + 2 - 10^1 = -7$$

Since $f(0) > 0$ and $f(1) < 0$, the Intermediate Value Theorem tells us that $f(c) = 0$ for some c in the interval $(0, 1)$, all of whose elements are positive numbers.