Math 180, Final Exam, Fall 2011
Problem 1 Solution

1. Calculate the following limits:

(a) $\displaystyle\lim_{x\to\frac{\pi}{2}} \frac{3\sin(x)\cos(x)}{\pi - 2x}$

(b) $\displaystyle\lim_{x\to\infty} \sqrt{x^2+x} - x$

(c) $\displaystyle\lim_{x\to 1} \frac{x^2+1}{x+4}$

Solution:

(a) Upon substituting $x = \frac{\pi}{2}$ we get the indeterminate form $\frac{0}{0}$. This indeterminacy is resolved using L'Hôpital's Rule.

$$\lim_{x\to\frac{\pi}{2}} \frac{3\sin(x)\cos(x)}{\pi - 2x} \overset{\text{L'H}}{=} \lim_{x\to\frac{\pi}{2}} \frac{\frac{d}{dx}\,3\sin(x)\cos(x)}{\frac{d}{dx}\,(\pi - 2x)},$$

$$\lim_{x\to\frac{\pi}{2}} \frac{3\sin(x)\cos(x)}{\pi - 2x} = \lim_{x\to\frac{\pi}{2}} \frac{3\cos(x)\cos(x) - 3\sin(x)\sin(x)}{-2},$$

$$\lim_{x\to\frac{\pi}{2}} \frac{3\sin(x)\cos(x)}{\pi - 2x} = \lim_{x\to\frac{\pi}{2}} \frac{3(\cos^2(x) - \sin^2(x))}{-2},$$

$$\lim_{x\to\frac{\pi}{2}} \frac{3\sin(x)\cos(x)}{\pi - 2x} = \frac{3(\cos^2(\frac{\pi}{2}) - \sin^2(\frac{\pi}{2}))}{-2},$$

$$\lim_{x\to\frac{\pi}{2}} \frac{3\sin(x)\cos(x)}{\pi - 2x} = \frac{3(0-1)}{-2},$$

$$\boxed{\lim_{x\to\frac{\pi}{2}} \frac{3\sin(x)\cos(x)}{\pi - 2x} = \frac{3}{2}}$$

(b) This limit has the indeterminate form $\infty - \infty$. To resolve this indeterminacy by multiplying the function by its conjugate divided by itself.

$$\lim_{x\to\infty} \sqrt{x^2+x} - x = \lim_{x\to\infty} \left(\sqrt{x^2+x} - x\right) \cdot \left(\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}\right),$$

$$\lim_{x\to\infty} \sqrt{x^2+x} - x = \lim_{x\to\infty} \frac{(x^2+x) - x^2}{\sqrt{x^2+x}+x},$$

$$\lim_{x\to\infty} \sqrt{x^2+x} - x = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x}.$$

This limit has the indeterminate form $\frac{\infty}{\infty}$ so it is a candidate for using L'Hôpital's rule. However, we will proceed by multiplying the function by $\frac{\frac{1}{x}}{\frac{1}{x}}$.

$$\lim_{x\to\infty} \sqrt{x^2+x} - x = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x},$$

$$\lim_{x\to\infty} \sqrt{x^2+x} - x = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}},$$

$$\lim_{x\to\infty} \sqrt{x^2+x} - x = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}+1},$$

$$\lim_{x\to\infty} \sqrt{x^2+x} - x = \frac{1}{\sqrt{1+0}+1},$$

$$\boxed{\lim_{x\to\infty} \sqrt{x^2+x} - x = \frac{1}{2}}$$

(c) The function is continuous at $x = 1$ so we can evaluate the limit by substitution.

$$\boxed{\lim_{x\to 1} \frac{x^2+1}{x+4} = \frac{1^2+1}{1+4} = \frac{2}{5}}$$

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Problem 2 Solution

2. Calculate the derivatives of the following functions:

(a) $\ln(x^2 + e^x)$

(b) $(x^2 - 5x + 1)\sqrt{x^4 + 10}$

(c) $\dfrac{3x + 2}{4x + 3}$

Solution:

(a) The derivative is obtained using the logarithm and chain rules.

$$\begin{aligned} \frac{d}{dx} \ln(x^2 + e^x) &= \frac{1}{x^2 + e^x} \cdot \frac{d}{dx}(x^2 + e^x) \\ &= \boxed{\frac{1}{x^2 + e^x} \cdot (2x + e^x)} \end{aligned}$$

(b) The derivative is obtained using the product and chain rules.

$$\begin{aligned} \frac{d}{dx} (x^2 - 5x + 1)\sqrt{x^4 + 10} &= (x^2 - 5x + 1)\frac{d}{dx}\sqrt{x^4 + 10} + \sqrt{x^4 + 10}\frac{d}{dx}(x^2 - 5x + 1), \\ &= (x^2 - 5x + 1) \cdot \frac{1}{2\sqrt{x^4 + 10}} \cdot \frac{d}{dx}(x^4 + 10) + \sqrt{x^4 + 10} \cdot (2x - 5), \\ &= \boxed{(x^2 - 5x + 1) \cdot \frac{1}{2\sqrt{x^4 + 10}} \cdot 4x^3 + \sqrt{x^4 + 10} \cdot (2x - 5)} \end{aligned}$$

(c) The derivative is obtained using the quotient rule.

$$\begin{aligned} \frac{d}{dx} \frac{3x + 2}{4x + 3} &= \frac{(4x + 3)\frac{d}{dx}(3x + 2) - (3x + 2)\frac{d}{dx}(4x + 3)}{(4x + 3)^2}, \\ &= \boxed{\frac{(4x + 3) \cdot 3 - (3x + 2) \cdot 4}{(4x + 3)^2}} \end{aligned}$$

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Problem 3 Solution

3. Let $f(x) = x^3 - 6x^2 + 9x - 5$.

(a) Find all critical points of f and classify them as local minima, local maxima, or neither.

(b) Find the intervals on which f is increasing or decreasing.

(c) Find the intervals on which f is concave up or concave down.

Solution:

(a) We begin by finding the critical points of f. The critical points of f are the values of x for which either $f'(x)$ does not exist or $f'(x) = 0$. Since $f(x)$ is a polynomial, we know that $f'(x)$ exists for all $x \in \mathbb{R}$ so the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0, \\ 3x^2 - 12x + 9 &= 0, \\ x^2 - 4x + 3 &= 0, \\ (x-1)(x-3) &= 0, \\ x = 1,\ x &= 3. \end{aligned}$$

We will use the First Derivative Test to classify each point. The domain of f is $(-\infty, \infty)$. We now split the domain into the three intervals $(-\infty, 1)$, $(1, 3)$, and $(3, \infty)$. We then evaluate $f'(x)$ at a test point in each interval to determine if there is a sign change in the first derivative.

Interval	**Test Point**, c	$f'(c)$	**Sign of** $f'(c)$
$(-\infty, 1)$	0	$f'(0) = 9$	$+$
$(1, 3)$	2	$f'(2) = -3$	$-$
$(3, \infty)$	4	$f'(4) = 9$	$+$

The first derivative changes sign from $+$ to $-$ across $x = 1$. Therefore, $x = 1$ corresponds to a local maximum. The first derivative changes sign from $-$ to $+$ across $x = 3$. Therefore, $x = 3$ corresponds to a local minimum.

(b) Using the table we conclude that f is increasing on $\boxed{(-\infty, 1) \cup (3, \infty)}$ because $f'(x) > 0$ for all x in these intervals and that f is decreasing on $\boxed{(1, 3)}$ because $f'(x) < 0$ for all x in this interval.

(c) To determine the intervals of concavity we start by finding solutions to the equation $f''(x) = 0$ and where $f''(x)$ does not exist. However, since $f(x)$ is a polynomial we know that $f''(x)$ will exist for all $x \in \mathbb{R}$. The solutions to $f''(x) = 0$ are:

$$\begin{aligned} f''(x) &= 0 \\ 6x - 12 &= 0, \\ x &= 2. \end{aligned}$$

We now split the domain into the two intervals $(-\infty, 2)$ and $(2, \infty)$. We then evaluate $f''(x)$ at a test point in each interval to determine the intervals of concavity.

Interval	**Test Point**, c	$f''(c)$	**Sign of** $f''(c)$
$(-\infty, 2)$	1	$f''(1) = -6$	$-$
$(2, \infty)$	3	$f''(3) = 6$	$+$

Using the table we conclude that f is concave up on $\boxed{(2, \infty)}$ because $f''(x) > 0$ for all x in this interval and that f is concave down on $\boxed{(-\infty, 2)}$ because $f''(x) < 0$ for all x in this interval.

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Problem 4 Solution

4. Let x and y be numbers in the interval $[1, 5]$ with $x + y = 6$.

(a) Determine the values of x and y which make xy^2 as large as possible.

(b) Determine the values of x and y which make xy^2 as small as possible.

Solution:

(a) Using the fact that $x + y = 6$ we have $x = 6 - y$. Therefore, the function for which we seek the absolute maximum value is $f(y) = (6 - y)y^2 = 6y^2 - y^3$. We begin by finding the critical points of f in the interior of the interval $[1, 5]$. These will be the values of y for which $f'(y) = 0$.

$$\begin{aligned} f'(y) &= 0, \\ 12y - 3y^2 &= 0, \\ 3y(4 - y) &= 0, \\ y = 0,\ y &= 4. \end{aligned}$$

Since $y = 0$ is outside the interval $[1, 5]$, the only critical point of interest is $y = 4$. We now evaluate $f(y)$ at $y = 4$ and at the endpoints of the interval, $y = 1$ and $y = 5$.

$$f(4) = 32, \quad f(1) = 5, \quad f(5) = 25$$

The largest of the above function values is 32 and it occurs when $\boxed{y = 4}$. The corresponding value of x is $\boxed{x = 2}$ since $x + y = 6$.

(b) The smallest of the function values computed in part (a) is 5 and it occurs at $\boxed{y = 1}$. The corresponding value of x is $\boxed{x = 5}$ since $x + y = 6$.

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Problem 5 Solution

5. Find an anti-derivative for each of the following functions:

(a) $2x^3 - 1$

(b) $e^x - e^{-x}$

(c) $\dfrac{2}{x+1}$

Solution:

(a) An antiderivative is found using the power rule.

$$\int (2x^3 - 1)\, dx = \frac{1}{2}x^4 - x + C$$

(b) An antiderivative is found using the exponential rule.

$$\int (e^x - e^{-x})\, dx = e^x + e^{-x} + C$$

(c) An antiderivative is found using the substitution $u = x + 1$, $du = dx$. We get

$$\int \frac{2}{x+1}\, dx = 2\int \frac{1}{u}\, du = 2\ln|u| = 2\ln|x+1| + C$$

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Problem 6 Solution

6. Calculate the following definite integrals.

(a) $\int_0^2 x\sqrt{2x^2+1}\,dx$

(b) $\int_0^{\pi/3} \sin(x)\cos(\pi\cos(x))\,dx$

Solution:

(a) We evaluate the integral using the substitution $u = 2x^2+1$, $\frac{1}{4}\,du = x\,dx$. The limits of integration become $u = 2(0)^2+1 = 1$ and $u = 2(2)^2+1 = 9$. Making the substitutions and evaluating the integral we get

$$\int_0^2 x\sqrt{2x^2+1}\,dx = \frac{1}{4}\int_1^9 \sqrt{u}\,du,$$

$$\int_0^2 x\sqrt{2x^2+1}\,dx = \frac{1}{4}\left[\frac{2}{3}u^{3/2}\right]_1^9,$$

$$\int_0^2 x\sqrt{2x^2+1}\,dx = \frac{1}{4}\left[\frac{2}{3}(9)^{3/2} - \frac{2}{3}(1)^{3/2}\right],$$

$$\boxed{\int_0^2 x\sqrt{2x^2+1}\,dx = \frac{13}{3}}$$

(b) We evaluate the integral using the substitution $u = \pi\cos(x)$, $-\frac{1}{\pi}\,du = \sin(x)\,dx$. The limits of integration become $u = \pi\cos(0) = \pi$ and $u = \pi\cos(\frac{\pi}{3}) = \frac{\pi}{2}$. Making the substitutions and evaluating the integral we get

$$\int_0^{\pi/3} \sin(x)\cos(\pi\cos(x))\,dx = -\frac{1}{\pi}\int_\pi^{\pi/2} \cos(u)\,du,$$

$$\int_0^{\pi/3} \sin(x)\cos(\pi\cos(x))\,dx = \frac{1}{\pi}\int_{\pi/2}^{\pi} \cos(u)\,du,$$

$$\int_0^{\pi/3} \sin(x)\cos(\pi\cos(x))\,dx = \frac{1}{\pi}\Big[\sin(u)\Big]_{\pi/2}^{\pi},$$

$$\int_0^{\pi/3} \sin(x)\cos(\pi\cos(x))\,dx = \frac{1}{\pi}\left[\sin\pi - \sin\frac{\pi}{2}\right],$$

$$\boxed{\int_0^{\pi/3} \sin(x)\cos(\pi\cos(x))\,dx = -\frac{1}{\pi}}$$

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Problem 7 Solution

7. Let $A(x)$ be the area below the curve $\sqrt{t^3+1}$ between $t=0$ and $t=x$.

(a) Express $A(x)$ as a definite integral.

(b) Calculate $A'(x)$.

(c) Use the linear approximation of $A(x)$ at $x=0$ to approximate $A(0.1)$.

Solution:

(a) The area function is

$$\boxed{A(x)=\int_0^x \sqrt{t^3+1}\,dt}$$

(b) Using the Fundamental Theorem of Calculus we get

$$\boxed{A'(x)=\sqrt{x^3+1}.}$$

(c) The linearization of $A(x)$ about $x=0$ is the function

$$L(x)=A(0)+A'(0)(x-0).$$

We know that $A(0)=0$ and that $A'(0)=\sqrt{0^3+1}=1$. Therefore, the linearization is

$$L(x)=x$$

The approximate value of $A(0.1)$ is $\boxed{L(0.1)=0.1}$.