Math 180, Final Exam, Fall 2012
Problem 1 Solution

1. Find the derivatives of the following functions:

(a) $\ln(\ln(x))$

(b) $x^6 + \sin(x) \cdot e^x$

(c) $\tan(x^2) + \cot(x^2)$

Solution:

(a) We evaluate the derivative using the Chain Rule.

$$\frac{d}{dx} \ln(\ln(x)) = \frac{1}{\ln(x)} \cdot \frac{d}{dx} \ln(x) = \frac{1}{\ln(x)} \cdot \frac{1}{x}$$

(b) We evaluate the derivative using the Power and Product Rules.

$$\frac{d}{dx} \left(x^6 + \sin(x) \cdot e^x\right) = 6x^5 + \sin(x) \cdot e^x + \cos(x) \cdot e^x$$

(c) We evaluate the derivative using the Chain Rule.

$$\frac{d}{dx} \left(\tan(x^2) + \cot(x^2)\right) = \sec^2(x^2) \cdot 2x - \csc^2(x^2) \cdot 2x$$

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Problem 2 Solution

2. Let $f(x) = 24x^3 - 48x + 3$.

(a) Find all local maxima and minima of $f(x)$.

(b) Find the absolute maximum and minimum of $f(x)$ on $[0, 2]$.

Solution:

(a) The function will attain local extreme values at its critical points, i.e. the values of x satisfying $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ 72x^2 - 48 &= 0 \\ 24(3x^2 - 2) &= 0 \\ x^2 &= \frac{2}{3} \\ x &= \pm\sqrt{\frac{2}{3}} \end{aligned}$$

To classify these points, we evaluate $f'(x)$ on either side of each critical point to determine how f' changes sign.

$$f'(-1) = 24, \quad f'(0) = -48, \quad f'(1) = 24$$

Since f' changes from positive to negative across $x = -\sqrt{\frac{2}{3}}$, the value of $f(-\sqrt{\frac{2}{3}})$ is a local maximum.

Since f' changes from negative to positive across $x = \sqrt{\frac{2}{3}}$, the value of $f(\sqrt{\frac{2}{3}})$ is a local minimum.

(b) f is continuous on $[0, 2]$ so we are guaranteed absolute extrema at either the endpoints or at a critical point in the interior of the interval. The critical points were computed in part (a) and only $x = \sqrt{\frac{2}{3}}$ lies in the given interval. The function values at $x = 0, \sqrt{\frac{2}{3}}, 2$ are

$$f(0) = 3, \quad f(\sqrt{\tfrac{2}{3}}) = -32\sqrt{\tfrac{2}{3}} + 3, \quad f(2) = 99$$

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Problem 3 Solution

3. A cylindrical cup of height h and radius r has volume $V = \pi r^2 h$ and surface area $\pi r^2 + \pi r h$. Among all such cups with volume $V = \pi$, find the one with minimal surface area.

Solution: The constraint in this problem is that the volume is constant. That is,

$$\begin{aligned} V &= \pi \\ \pi r^2 h &= \pi \\ h &= \frac{1}{r^2} \end{aligned}$$

The function we want to minimize is the surface area. Using the above equation, we can write the surface area as a function of r only.

$$\begin{aligned} f(r) &= \pi r^2 + \pi r h \\ f(r) &= \pi r^2 + \pi r \cdot \frac{1}{r^2} \\ f(r) &= \pi \left(r^2 + \frac{1}{r} \right), \quad r > 0 \end{aligned}$$

The critical points of f are:

$$\begin{aligned} f'(r) &= 0 \\ \pi \left(2r - \frac{1}{r^2} \right) &= 0 \\ 2r &= \frac{1}{r^2} \\ r^3 &= \frac{1}{2} \\ r &= \frac{1}{\sqrt[3]{2}} \end{aligned}$$

The second derivative of f is

$$f'(r) = \pi \left(2 + \frac{2}{r^3} \right)$$

and is positive for all $r > 0$. Thus, the function is concave up on $(0, \infty)$ and $r = \frac{1}{\sqrt[3]{2}}$ corresponds to an absolute minimum of f. The corresponding height of the cup is

$$h = \frac{1}{r^2} = \frac{1}{2^{2/3}}$$

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Problem 4 Solution

4. Determine the following limits

(a) $\lim_{x\to 0} \frac{\sin^2(x)}{x^2}$

(b) $\lim_{x\to 0^+} \frac{1}{(\ln(x))^2}$

(c) $\lim_{x\to 1} \frac{\sin^2(\pi x)}{x+1}$

Solution:

(a) The value of the limit is

$$\lim_{x\to 0} \frac{\sin^2(x)}{x^2} = \left(\lim_{x\to 0} \frac{\sin(x)}{x}\right)^2 = 1^2 = 1$$

(b) Since $\ln(x) \to -\infty$ as $x \to 0^+$, the value of the limit is 0.

(c) The given function is continuous at all $x \neq -1$. Therefore, we may use substitution.

$$\lim_{x\to 1} \frac{\sin^2(\pi x)}{x+1} = \frac{\sin^2(\pi)}{1+1} = 0$$

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Problem 5 Solution

5. Evaluate the following definite integrals.

(a) $\displaystyle\int_1^2 (\sqrt{x} + \sqrt{1+x})\,dx$

(b) $\displaystyle\int_0^{\pi/2} \sin(x)\sqrt{1-\cos(x)}\,dx$

Solution:

(a) Using the Fundamental Theorem of Calculus we obtain:

$$\begin{aligned}\int_1^2 (\sqrt{x} + \sqrt{1+x})\,dx &= \left[\frac{2}{3}x^{3/2} + \frac{2}{3}(1+x)^{3/2}\right]_1^2 \\ &= \left[\frac{2}{3}\cdot 2^{3/2} + \frac{2}{3}\cdot 3^{3/2}\right] - \left[\frac{2}{3} + \frac{2}{3}\cdot 2^{3/2}\right] \\ &= \frac{2}{3}\cdot 3^{3/2} - \frac{2}{3} \\ &= \frac{2}{3}(3\sqrt{3}-1)\end{aligned}$$

(b) Our strategy here is to let $u = 1-\cos(x)$, $du = \sin(x)\,dx$. The limits of integration change to $x = 1-\cos(0) = 0$ and $x = 1-\cos(\pi/2) = 1$. Upon making these substitutions and using the Fundamental Theorem of Calculus we have

$$\begin{aligned}\int_0^{\pi/2} \sin(x)\sqrt{1-\cos(x)}\,dx &= \int_0^1 \sqrt{u}\,du \\ &= \left[\frac{2}{3}u^{3/2}\right]_0^1 \\ &= \frac{2}{3}\end{aligned}$$

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Problem 6 Solution

6. Let $g(x) = x^2 - x + 1$.

(a) Using only the definition of the derivative, determine the value of $g'(1)$.

(b) Find the equation of the line tangent to the graph of $g(x)$ at $(2, 3)$.

Solution:

(a) The value of $g'(1)$ is

$$\begin{aligned} g'(1) &= \lim_{x\to 1} \frac{g(x) - g(1)}{x - 1} \\ &= \lim_{x\to 1} \frac{(x^2 - x + 1) - 1}{x - 1} \\ &= \lim_{x\to 1} \frac{x^2 - x}{x - 1} \\ &= \lim_{x\to 1} \frac{x(x - 1)}{x - 1} \\ &= \lim_{x\to 1} x \\ &= 1 \end{aligned}$$

(b) The derivative of $g(x)$ is $g'(x) = 2x - 1$. Thus, the slope of the tangent line at $(2, 3)$ is $g'(2) = 2(2) - 1 = 3$. Therefore, the equation of the tangent line is

$$y - 3 = 3(x - 2)$$

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Problem 7 Solution

7. For some number c, we define the function $h(x)$ by $h(x) = x^2 + 1$ if $x \geq 2$ and by $h(x) = \dfrac{1}{3-x} + c$ if $x < 2$.

(a) Determine $\lim_{x \to 2^+} h(x)$ and $\lim_{x \to 2^-} h(x)$.

(b) For which value or values of c does $\lim_{x \to 2} h(x)$ exist?

(c) For each of the values of c computed in part (b), determine whether or not $h(x)$ is differentiable at $x = 2$.

Solution:

(a) The one-sided limits are

$$\lim_{x \to 2^+} h(x) = 2^2 + 1 = 5$$
$$\lim_{x \to 2^-} h(x) = \lim_{x \to 2^-} \frac{1}{3-x} + c = \frac{1}{3-2} + c = 1 + c$$

(b) The limit exists when the one-sided limits are the same. This occurs when

$$1 + c = 5 \iff c = 4$$

(c) The derivative of $h(x)$ is $2x$ if $x > 2$ and $\dfrac{1}{(3-x)^2}$ when $x < 2$. The derivative approaches $2(2) = 4$ as $x \to 2^+$ and $\dfrac{1}{(3-2)^2} = 1$ as $x \to 2^-$. Since these limits are not the same, the function is not differentiable at $x = 2$.

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Problem 8 Solution

8. Determine $\lim_{x\to\infty} f(x)$ for each of the following functions.

(a) $f(x) = \dfrac{2}{x-3}$

(b) $f(x) = \dfrac{x^3 - \sqrt{x}}{2x^2 - \sqrt{x}}$

Solution:

(a) $\lim_{x\to\infty} \dfrac{2}{x-3} = 0$ since $f(x)$ is a rational function where

$$\deg(p(x)) < \deg(q(x))$$

($p(x)$ and $q(x)$ are the numerator and denominator of $f(x)$, respectively).

(b) The limit is computed as follows:

$$\begin{aligned}\lim_{x\to\infty} \frac{x^3 - \sqrt{x}}{2x^2 - \sqrt{x}} &= \lim_{x\to\infty} \frac{x^3 - \sqrt{x}}{2x^2 - \sqrt{x}} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\ &= \lim_{x\to\infty} \frac{x - \frac{1}{x^{3/2}}}{2 - \frac{1}{x^{3/2}}} \quad \left(\to \frac{\infty - 0}{2 - 0}\right) \\ &= \infty\end{aligned}$$

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Problem 9 Solution

9. Let $f(x) = \dfrac{2x}{x^2 - 4}$.

(a) Find all horizontal and vertical asymptotes of $f(x)$.

(b) Find the area of the region bounded by the x-axis, the line $x = 3$, the line $x = 4$, and the graph of $f(x)$.

Solution:

(a) Since

$$\lim_{x \to 2^+} \frac{2x}{x^2 - 4} = +\infty$$

we know that $x = 2$ is a vertical asymptote of $f(x)$. Furthermore, since

$$\lim_{x \to -2^+} \frac{2x}{x^2 - 4} = +\infty$$

we know that $x = -2$ is also a vertical asymptote.

Since

$$\lim_{x \to \pm\infty} \frac{2x}{x^2 - 4} = 0$$

we know that $y = 0$ is a horizontal asymptote of $f(x)$.

(b) The area of the region is

$$\begin{aligned}
A &= \int_3^4 \frac{2x}{x^2 - 4}\, dx \\
&= \Big[\ln(x^2 - 4) \Big]_3^4 \\
&= \ln(4^2 - 4) - \ln(3^2 - 4) \\
&= \ln(12) - \ln(5) \\
&= \ln \frac{12}{5}
\end{aligned}$$