Math 180, Final Exam, Spring 2010
Problem 1 Solution

1. Differentiate the following functions:

(a) $f(x) = x^2 \tan(3x - 2)$

(b) $f(x) = (x^2 + 1)^3$

(c) $f(x) = \ln(5 + 2\cos x)$

Solution:

(a) Use the Product and Chain Rules.

$$\begin{aligned} f'(x) &= [x^2 \tan(3x-2)]' \\ &= x^2[\tan(3x-2)]' + (x^2)' \tan(3x-2) \\ &= x^2 \sec^2(3x-2) \cdot (3x-2)' + 2x \tan(3x-2) \\ &= \boxed{x^2 \sec^2(3x-2) \cdot (3) + 2x \tan(3x-2)} \end{aligned}$$

(b) Use the Chain Rule.

$$\begin{aligned} f'(x) &= [(x^2+1)^3]' \\ &= 3(x^2+1)^2 \cdot (x^2+1)' \\ &= \boxed{3(x^2+1)^2 \cdot (2x)} \end{aligned}$$

(c) Use the Chain Rule.

$$\begin{aligned} f'(x) &= [\ln(5 + 2\cos x)]' \\ &= \frac{1}{5 + 2\cos x} \cdot (5 + 2\cos x)' \\ &= \boxed{\frac{1}{5 + 2\cos x} \cdot (-2\sin x)} \end{aligned}$$

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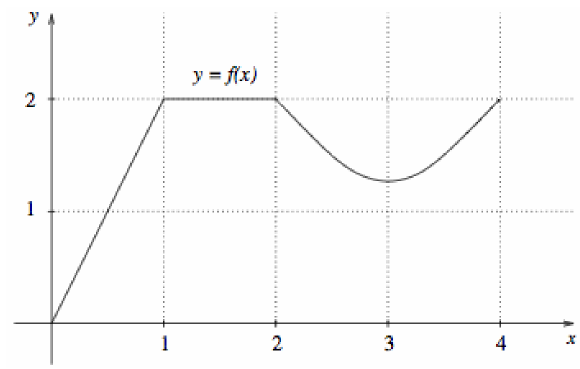
Problem 2 Solution

2. Let $f(x)$ be the function whose graph is shown below.

(a) Compute the average rate of change of $f(x)$ over the interval $[0, 4]$.

(b) Compute $f'(0.5)$, $f'(1.5)$, and $f'(3)$.

(c) Compute $\int_0^2 f(x)\,dx$.



Solution:

(a) The average rate of change formula is:

$$\text{average ROC} = \frac{f(b) - f(a)}{b - a}$$

Using the graph and the values $a = 0$, $b = 4$, we get:

$$\text{average ROC} = \frac{f(4) - f(0)}{4 - 0} = \frac{2 - 0}{4} = \boxed{\frac{1}{2}}$$

(b) The derivative $f'(x)$ represents the slope of the line tangent to the graph at x. From the graph we find that:

$$f'(0.5) = 2, \quad f'(1.5) = 0, \quad f'(3) = 0$$

(c) The value of $\int_0^2 f(x)\,dx$ represents the signed area between the curve $y = f(x)$ and the x-axis. Using geometry, we find the value of the integral by adding the area of the triangle (the region below $y = f(x)$ on the interval $[0, 1]$) to the area of the rectangle (the region below $y = f(x)$ on the interval $[1, 2]$).

$$\int_0^2 f(x)\,dx = \text{triangle area} + \text{rectangle area} = \frac{1}{2}(1)(2) + (1)(2) = \boxed{3}$$

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Problem 3 Solution

3. Let $f(x) = 3x^4 - 4x^3 + 1$.

(a) Find and classify the critical point(s) of f.

(b) Determine the intervals where f is increasing and where f is decreasing.

(c) Find the inflection point(s) of f.

(d) Determine the intervals where f is concave up and where f is concave down.

Solution:

(a) The critical points of $f(x)$ are the values of x for which either $f'(x)$ does not exist or $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(3x^4 - 4x^3 + 1\right)' &= 0 \\ 12x^3 - 12x^2 &= 0 \\ 12x^2(x-1) &= 0 \\ x = 0,\ x &= 1 \end{aligned}$$

Thus, $\boxed{x = 0}$ and $\boxed{x = 1}$ are the critical points of f.

The domain of f is $(-\infty, \infty)$. We now split the domain into the intervals $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$. We then evaluate $f'(x)$ at a test point in each interval to determine the intervals of monotonicity.

Interval	**Test Point**, c	$f'(c)$	**Sign of** $f'(c)$
$(-\infty, 0)$	-1	$f'(-1) = -24$	$-$
$(0, 1)$	$\frac{1}{2}$	$f'(\frac{1}{2}) = -\frac{3}{2}$	$-$
$(1, \infty)$	2	$f'(2) = 24$	$+$

Since f' changes sign from $-$ to $+$ at $x = 1$, the First Derivative Test implies that $f(1) = 0$ is a local minimum. However, since f' does not change sign at $x = 0$, $f(0) = 1$ is neither a local minimum nor a local maximum.

(b) Using the table above, we conclude that f is increasing on $(1, \infty)$ because $f'(x) > 0$ for all $x \in (1, \infty)$ and f is decreasing on $(-\infty, 0) \cup (0, 1)$ because $f'(x) < 0$ for all $x \in (-\infty, 0) \cup (0, 1)$.

(c) To determine the intervals of concavity we start by finding solutions to the equation $f''(x) = 0$ and where $f''(x)$ does not exist.

$$\begin{aligned} f''(x) &= 0 \\ \left(12x^3 - 12x^2\right)' &= 0 \\ 36x^2 - 24x &= 0 \\ 12x(3x - 2) &= 0 \\ x = 0,\ x &= \frac{2}{3} \end{aligned}$$

We now split the domain into the intervals $(-\infty, 0)$, $(0, \frac{2}{3})$, and $(\frac{2}{3}, \infty)$. We then evaluate $f''(x)$ at a test point in each interval to determine the intervals of concavity.

Interval	**Test Point**, c	$f''(c)$	**Sign of** $f''(c)$
$(-\infty, 0)$	-1	$f''(-1) = 60$	$+$
$(0, \frac{2}{3})$	$\frac{1}{2}$	$f''(\frac{1}{2}) = -3$	$-$
$(\frac{2}{3}, \infty)$	1	$f''(1) = 12$	$+$

The inflection points of $f(x)$ are the points where $f''(x)$ changes sign. We can see in the above table that $f''(x)$ changes sign at $x = 0$ and $x = \frac{2}{3}$. Therefore, $\boxed{x = 0,\ \frac{2}{3}}$ are inflection points.

(d) Using the table above, we conclude that f is concave down on $(0, \frac{2}{3})$ because $f''(x) < 0$ for all $x \in (0, \frac{2}{3})$ and f is concave up on $(-\infty, 0) \cup (\frac{2}{3}, \infty)$ because $f''(x) > 0$ for all $x \in (-\infty, 0) \cup (\frac{2}{3}, \infty)$.

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Problem 4 Solution

4. Find the equation of the tangent line to the curve $xy + x^2y^2 = 6$ at the point $(2, 1)$.

Solution: We must find $\frac{dy}{dx}$ using implicit differentiation.

$$\begin{aligned}
xy + x^2y^2 &= 6 \\
\frac{d}{dx}(xy) + \frac{d}{dx}(x^2y^2) &= \frac{d}{dx}6 \\
\left(x\frac{dy}{dx} + y\right) + \left(2x^2y\frac{dy}{dx} + 2xy^2\right) &= 0 \\
x\frac{dy}{dx} + 2x^2y\frac{dy}{dx} &= -y - 2xy^2 \\
\frac{dy}{dx}\left(x + 2x^2y\right) &= -y - 2xy^2 \\
\frac{dy}{dx} &= \frac{-y - 2xy^2}{x + 2x^2y}
\end{aligned}$$

The value of $\frac{dy}{dx}$ at $(2, 1)$ is the slope of the tangent line.

$$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{-1 - 2(2)(1)^2}{2 + 2(2)^2(1)} = -\frac{1}{2}$$

An equation for the tangent line at $(2, 1)$ is then:

$$\boxed{y - 1 = -\frac{1}{2}(x - 2)}$$

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Problem 5 Solution

5. Use L'Hôpital's rule to compute: $\lim_{x\to 0} \frac{x^2+x}{\sin x}$.

Solution: Upon substituting $x = 0$ into the function we find that

$$\frac{x^2+x}{\sin x} = \frac{0^2+0}{\sin 0} = \frac{0}{0}$$

which is indeterminate. We resolve this indeterminacy by using L'Hôpital's Rule.

$$\begin{aligned}\lim_{x\to 0} \frac{x^2+x}{\sin x} &\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{(x^2+x)'}{(\sin x)'} \\ &= \lim_{x\to 0} \frac{2x+1}{\cos x} \\ &= \frac{2(0)+1}{\cos 0} \\ &= \boxed{1}\end{aligned}$$

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Problem 6 Solution

6. Compute the indefinite integrals:

(a) $\displaystyle\int \frac{x-1}{\sqrt{x}}\,dx$

(b) $\displaystyle\int (\sin x - \cos(3x))\,dx$

Solution:

(a) $$\int \frac{x-1}{\sqrt{x}}\,dx = \int \left(\frac{x}{\sqrt{x}} - \frac{1}{\sqrt{x}}\right)\,dx = \int \left(x^{1/2} - x^{-1/2}\right)\,dx = \boxed{\frac{2}{3}x^{3/2} - 2x^{1/2} + C}$$

(b) $$\int (\sin x - \cos(3x))\,dx = \boxed{-\cos x - \frac{1}{3}\sin(3x) + C}$$

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Problem 7 Solution

7.

(a) Write the integral which gives the area of the region between $x = 0$ and $x = 2$, below the curve $y = 1 - e^{-x}$, and above the x axis.

(b) Evaluate the integral exactly to find the area.

Solution:

(a) The area of the region is given by the integral:

$$\int_0^2 (1 - e^{-x})\, dx$$

(b) We use FTC I to evaluate the integral.

$$\begin{aligned} \int_0^2 (1 - e^{-x})\, dx &= x + e^{-x}\Big|_0^2 \\ &= (2 + e^{-2}) - (0 + e^{-0}) \\ &= \boxed{1 + e^{-2}} \end{aligned}$$

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Problem 8 Solution

8. Show that there is a positive real solution of the equation $x^7 = x + 1$.

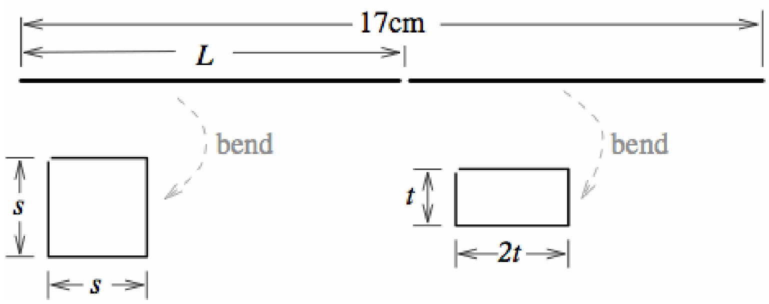
Solution: Let $f(x) = x^7 - x - 1$. First we recognize that $f(x)$ is continuous everywhere. Next, we must find an interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs. Let's choose $a = 0$ and $b = 2$.

$$f(0) = 0^7 - 0 - 1 = -1$$
$$f(2) = 2^7 - 2 - 1 = 125$$

Since $f(0) < 0$ and $f(2) > 0$, the Intermediate Value Theorem tells us that $f(c) = 0$ for some c in the interval $(0, 2)$, all of whose elements are positive numbers.

Math 180, Final Exam, Study Guide
Problem 10 Solution

10. A piece of wire of length 17 centimeters will be cut into two pieces. The first piece of length L will be bent into a square, while the rest of the wire will be bent into a rectangle whose width is twice its height. Determine the length L that will minimize the sum of the areas enclosed by the square and the rectangle. For this value of L, also determine the total enclosed area, the side length s of the square, and the length t of the shorter side of the rectangle.



Solution: First, we recognize that:

$$L = 4s, \qquad 17 - L = 6t$$

using the fact that the right hand sides of the equations represent the perimeter of the square and rectangle, respectively. The sum of the areas is:

$$A = s^2 + 2t^2$$

Let's make A a function of L by solving the first two equations for s and t, respectively, and plugging them into the third.

$$s = \frac{L}{4}, \quad t = \frac{17 - L}{6}$$

The sum of the areas is then

$$A(L) = \left(\frac{L}{4}\right)^2 + 2\left(\frac{17 - L}{6}\right)^2, \quad 0 \leq L \leq 17$$

There is only one critical point of $A(L)$ and it is the solution to $A'(L) = 0$.

$$\begin{aligned} A'(L) &= 0 \\ 2\left(\frac{L}{4}\right) \cdot \frac{1}{4} + 4\left(\frac{17 - L}{6}\right) \cdot \left(-\frac{1}{6}\right) &= 0 \\ \frac{1}{16}L - \frac{1}{18}(17 - L) &= 0 \\ \frac{9}{8}L - 17 + L &= 0 \\ \frac{17}{8}L &= 17 \\ L &= 8 \end{aligned}$$

The corresponding values of s and t are then

$$s = \frac{L}{4} = 2, \qquad t = \frac{17 - L}{6} = \frac{3}{2}$$

and the sum of the areas is

$$A(8) = \left(\frac{8}{4}\right)^2 + 2\left(\frac{17-8}{6}\right)^2 = \frac{17}{2} = \frac{17^2}{34}$$

To ensure that the value of L above corresponds to a minimum, we check the endpoints of the domain of definition of $A(L)$.

$$A(0) = 2\left(\frac{17}{6}\right)^2 = \frac{17^2}{18}, \qquad A(17) = \left(\frac{17}{4}\right)^2 = \frac{17^2}{16}$$

We recognize that

$$\frac{17^2}{34} < \frac{17^2}{18} < \frac{17^2}{16}$$

Clearly, $A(8)$ is the minimum value of $A(L)$ on the domain of definition.