Math 180, Final Exam, Spring 2011
Problem 1 Solution

1. Evaluate the following limits, or show that they do not exist.

(a) $\displaystyle\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$

(b) $\displaystyle\lim_{x\to 0}\frac{|x^2-4|}{3x+6}$

Solution:

(a) The given limit is, by definition, the derivative of the function e^x. Thus,

$$\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}=\frac{d}{dx}e^x=\boxed{e^x}$$

(b) The function $f(x)=\dfrac{|x^2-4|}{3x+6}$ is continuous at $x=0$. Thus, we evaluate the limit using the substitution method.

$$\lim_{x\to 0}\frac{|x^2-4|}{3x+6}=\frac{|0^2-4|}{3(0)+6}=\boxed{\frac{2}{3}}$$

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Problem 2 Solution

2. Differentiate the following functions. Leave your answers in unsimplified form so it is clear what method you used.

(a) $\dfrac{x}{2+\sin(x)}$

(b) $\arctan(e^x)$

(c) $\ln(e^x+1)$

Solution:

(a) Use the Quotient Rule.

$$\left(\frac{x}{2+\sin(x)}\right)' = \frac{[2+\sin(x)](x)' - (x)[2+\sin(x)]'}{[2+\sin(x)]^2}$$
$$= \frac{[2+\sin(x)](1) - (x)[\cos(x)]}{[2+\sin(x)]^2}$$
$$= \boxed{\frac{2+\sin(x) - x\cos(x)}{[2+\sin(x)]^2}}$$

(b) Use the Chain Rule.

$$[\arctan(e^x)]' = \frac{1}{1+(e^x)^2}\cdot(e^x)'$$
$$= \boxed{\frac{1}{1+e^{2x}}\cdot e^x}$$

(c) Use the Chain Rule.

$$[\ln(e^x+1)]' = \frac{1}{e^x+1}\cdot(e^x+1)'$$
$$= \boxed{\frac{1}{e^x+1}\cdot e^x}$$

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Problem 3 Solution

3. Calculate the indefinite integrals.

(a) $\displaystyle\int \frac{e^{-\frac{1}{x}}\,dx}{x^2}$

(b) $\displaystyle\int x\sin(x^2)\cos(x^2)dx$

Solution:

(a) We use the substitution $u = -\frac{1}{x}$, $du = \frac{1}{x^2}\,dx$. Making the substitutions and evaluating the integral we get:

$$\begin{aligned}\int \left(\frac{e^{-\frac{1}{x}}}{x^2}\right) dx &= \int e^{-\frac{1}{x}} \cdot \frac{1}{x^2}\,dx \\ &= \int e^u \cdot du \\ &= e^u + C \\ &= \boxed{e^{-\frac{1}{x}} + C}\end{aligned}$$

(b) We use the substitution $u = \sin(x^2)$, $\frac{1}{2}\,du = x\cos(x^2)\,dx$. Making the substitutions and evaluating the integral we get:

$$\begin{aligned}\int x\sin(x^2)\cos(x^2)\,dx &= \frac{1}{2}\int u\,du \\ &= \frac{1}{4}u^2 + C \\ &= \boxed{\frac{1}{4}\left[\sin(x^2)\right]^2 + C}\end{aligned}$$

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Problem 4 Solution

4. Calculate the definite integrals.

(a) $\displaystyle\int_0^{\pi/4} \sec^2(x)dx$

(b) $\displaystyle\int_0^2 \frac{x}{(x^2+2)^2}dx$

Solution:

(a) We use the Fundamental Theorem of Calculus, Part I to evaluate the integral.

$$\begin{aligned}\int_0^{\pi/4} \sec^2(x)dx &= \Big[\tan(x)\Big]_0^{\pi/4} \\ &= \tan\left(\frac{\pi}{4}\right) - \tan(0) \\ &= \boxed{1}\end{aligned}$$

(b) We use the u-substitution method and the Fundamental Theorem of Calculus, Part I to evaluate the integral. Let $u = x^2 + 2$, $\frac{1}{2}\,du = x\,dx$. The limits of integration then become $u = 0^2 + 2 = 2$ and $u = 2^2 + 2 = 6$. Making these substitutions and evaluating the integral, we get:

$$\begin{aligned}\int_0^2 \frac{x}{(x^2+2)^2}dx &= \frac{1}{2}\int_2^6 \frac{1}{u^2}\,du \\ &= \frac{1}{2}\left[-\frac{1}{u}\right]_2^6 \\ &= \frac{1}{2}\left[-\frac{1}{6} - \left(-\frac{1}{2}\right)\right] \\ &= \boxed{\frac{1}{6}}\end{aligned}$$

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Problem 5 Solution

5. Let $F(x) = \int_1^{x^2} \ln(t)\,dt$.

(a) Compute $F(-1)$.

(b) Find the derivative $F'(x)$.

Solution:

(a) The value of $F(-1)$ is:

$$F(-1) = \int_1^{(-1)^2} \ln(t)\,dt = \int_1^1 \ln(t)\,dt = \boxed{0}$$

(b) We use the Fundamental Theorem of Calculus, Part II and the Chain Rule to find $F'(x)$.

$$\begin{aligned} F'(x) &= \frac{d}{dx}\int_1^{x^2} \ln(t)\,dt \\ &= \ln\left(x^2\right)\cdot\frac{d}{dx}\left(x^2\right) \\ &= \boxed{\ln\left(x^2\right)\cdot(2x)} \end{aligned}$$

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Problem 6 Solution

6. Consider the function $f(x) = \frac{\ln(x)}{x}$ defined for $x > 0$.

(a) Find the critical point(s) of f.

(b) Classify each critical point as a local minimum, local maximum, or neither. Justify your answers.

(c) Find the absolute minimum and absolute maximum of f on the interval $[1, e^2]$.

Solution:

(a) The critical points of $f(x)$ are the values of x for which either $f'(x) = 0$ or $f'(x)$ does not exist. The derivative $f'(x)$ can be found using the quotient rule.

$$\begin{aligned} f'(x) &= \left(\frac{\ln(x)}{x}\right)' \\ &= \frac{(x)[\ln(x)]' - \ln(x)(x)'}{x^2} \\ &= \frac{x(\frac{1}{x}) - \ln(x)}{x^2} \\ &= \frac{1 - \ln(x)}{x^2} \end{aligned}$$

$f'(x)$ exists for all $x > 0$, which is the domain of f. Therefore, the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \frac{1 - \ln(x)}{x^2} &= 0 \\ 1 - \ln(x) &= 0 \\ \ln(x) &= 1 \\ x &= e \end{aligned}$$

The corresponding function value is $f(e) = \frac{\ln(e)}{e} = \frac{1}{e}$. Thus, the critical point is $(e, \frac{1}{e})$.

(b) We use the Second Derivative Test to classify the critical point. The second derivative is found using the quotient rule.

$$\begin{aligned} f''(x) &= \left(\frac{1 - \ln(x)}{x^2}\right)' \\ &= \frac{(x^2)[1 - \ln(x)]' - [1 - \ln(x)](x^2)'}{(x^2)^2} \\ &= \frac{(x^2)(-\frac{1}{x}) - [1 - \ln(x)](2x)}{x^4} \\ &= \frac{-3x + 2x\ln(x)}{x^4} \end{aligned}$$

At the critical point, we have:

$$f''(e) = \frac{-3e + 2e\ln(e)}{e^4} = -\frac{1}{e^3}$$

Since $f''(e) < 0$ the Second Derivative Test implies that $f(e) = \frac{1}{e}$ is a local maximum.

(c) The absolute extrema of f will occur either at a critical point in $[1, e^2]$ or at one of the endpoints. From part (a), we found that the critical number of f is $x = e$. Thus, we evaluate f at $x = 1$, $x = e$, and $x = e^2$.

$$f(1) = \frac{\ln(1)}{1} = 0$$
$$f(e) = \frac{\ln(e)}{e} = \frac{1}{e}$$
$$f(e^2) = \frac{\ln(e^2)}{e^2} = \frac{2}{e^2}$$

The absolute minimum of f on $[1, e^2]$ is $\boxed{0}$ because it is the smallest of the values of f above and the absolute maximum is $\boxed{\frac{1}{e}}$ because it is the largest.

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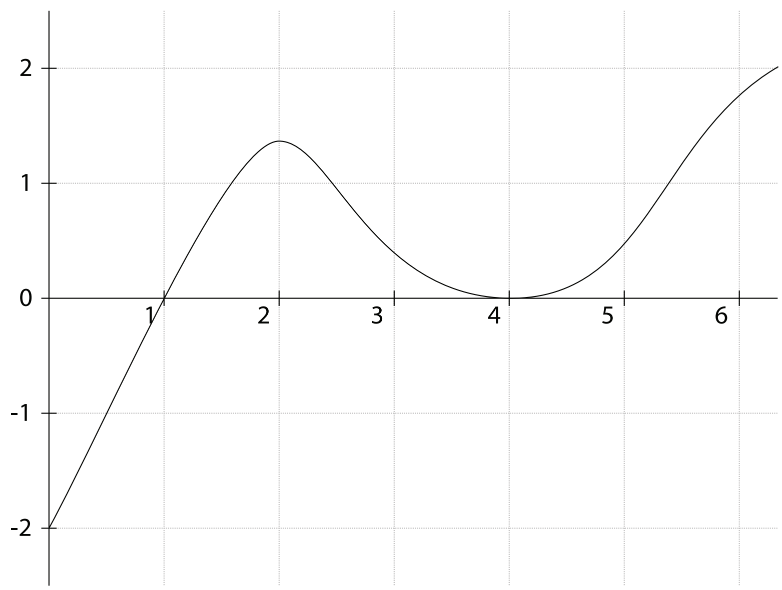
Problem 7 Solution

7. Shown below is the graph of $f'(x)$, the **derivative** of the function $f(x)$.

(a) Using the graph of $f'(x)$ below, determine the intervals where f is increasing, decreasing, concave up, and concave down.

(b) Given that $f(0) = 1$, sketch the graph of the function $f(x)$. On your graph, clearly label all maxima, minima, and inflection points.

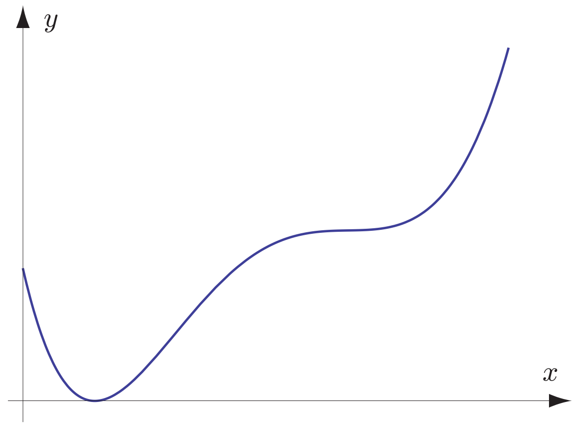


This is the graph of the **derivative** of $f(x)$.

Solution:

(a) $f(x)$ is increasing on $(1, 4) \cup (4, 6)$ because $f'(x) > 0$ for these values of x. $f(x)$ is decreasing on $(0, 1)$ because $f'(x) < 0$ for these values of x. $f(x)$ is concave up on $(0, 2) \cup (4, 6.5)$ because $f'(x)$ is increasing for these values of x. $f(x)$ is concave down on $(2, 4)$ because $f'(x)$ is decreasing for these values of x.

(b) $f(x)$ has a local minimum at $x = 1$ because $f'(1) = 0$ and $f'(x)$ changes from negative to positive at $x = 1$. $f(x)$ has an inflection point at $x = 2$ and $x = 4$ because $f''(x) = 0$ and $f''(x)$ changes sign at these values of x. A rough sketch of $f(x)$ is shown below. (Note: The graph is not necessarily to scale.)


y
x

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Problem 8 Solution

8. Consider the curve defined by the equation $x^3 + y^3 = x^2 + y^2$.

(a) Use implicit differentiation to find the derivative $\frac{dy}{dx}$ in terms of x and y.

(b) Find an equation for the tangent line to this curve at the point $(\frac{5}{9}, \frac{10}{9})$.

Solution:

(a) To find $\frac{dy}{dx}$, we use implicit differentiation.

$$\begin{aligned} x^3 + y^3 &= x^2 + y^2 \\ \frac{d}{dx}x^3 + \frac{d}{dx}y^3 &= \frac{d}{dx}x^2 + \frac{d}{dx}y^2 \\ 3x^2 + 3y^2\frac{dy}{dx} &= 2x + 2y\frac{dy}{dx} \\ 3y^2\frac{dy}{dx} - 2y\frac{dy}{dx} &= 2x - 3x^2 \\ \frac{dy}{dx}\left(3y^2 - 2y\right) &= 2x - 3x^2 \\ \frac{dy}{dx} &= \frac{2x - 3x^2}{3y^2 - 2y} \\ \frac{dy}{dx} &= \boxed{\frac{x}{y} \cdot \frac{2 - 3x}{3y - 2}} \end{aligned}$$

(b) The value of $\frac{dy}{dx}$ at $(\frac{5}{9}, \frac{10}{9})$ is the slope of the tangent line at the point $(\frac{5}{9}, \frac{10}{9})$.

$$\left.\frac{dy}{dx}\right|_{(\frac{5}{9}, \frac{10}{9})} = \frac{\frac{5}{9}}{\frac{10}{9}} \cdot \frac{2 - 3(\frac{5}{9})}{3(\frac{10}{9}) - 2} = \frac{1}{2} \cdot \frac{3}{12} = \frac{1}{8}$$

An equation for the tangent line is then:

$$\boxed{y - \frac{10}{9} = \frac{1}{8}\left(x - \frac{5}{9}\right)}$$

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Problem 9 Solution

9. A function g is defined on the interval $[1,3]$ by $g(x) = x^4 - x + 1$. Let $h(x) = g^{-1}(x)$ be the inverse function. Compute $h'(15)$.

Solution: The value of $h'(15)$ is given by the formula:

$$h'(15) = \frac{1}{g'(h(15))}$$

It isn't necessary to find a formula for $h(x)$ to find $h(15)$. We will use the fact that $g(2) = 2^4 - 2 + 1 = 15$ to say that $h(15) = 2$ by the property of inverses. The derivative of $g(x)$ is $g'(x) = 4x^3 - 1$. Therefore,

$$\begin{aligned} h'(15) &= \frac{1}{g'(h(15))} \\ &= \frac{1}{g'(2)} \\ &= \frac{1}{4(2)^3 - 1} \\ &= \boxed{\frac{1}{31}} \end{aligned}$$