Math 180, Final Exam, Spring 2012
Problem 1 Solution

1. Compute each limit or explain why it does not exist.

(a) $\displaystyle\lim_{x\to 3} \frac{1}{x^2-1}$

(b) $\displaystyle\lim_{x\to\infty} \frac{\sin(x)}{x}$

(c) $\displaystyle\lim_{x\to 2} \frac{\sqrt{3x+10}-4}{x-2}$

Solution:

(a) The function $f(x) = \dfrac{1}{x^2-1}$ is continuous at $x=3$. Therefore, we may evaluate the limit via substitution.

$$\boxed{\lim_{x\to 3} \frac{1}{x^2-1} = \frac{1}{3^2-1} = \frac{1}{8}.}$$

(b) The limit must be evaluated using the Squeeze Theorem. First, we note that

$$-\frac{1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}$$

for all $x > 0$. Furthermore, we know that

$$\lim_{x\to\infty}\left(-\frac{1}{x}\right) = \lim_{x\to\infty}\frac{1}{x} = 0.$$

Therefore, by the Squeeze Theorem, we have

$$\boxed{\lim_{x\to\infty} \frac{\sin(x)}{x} = 0.}$$

(c) Upon substituting $x=2$ we find that the limit is of the form $\frac{0}{0}$ which is indeterminate. We have two options: (1) use an algebraic method or (2) use L'Hopital's Rule. We will use an algebraic method, i.e. multiply by the conjugate divided by itself.

$$\lim_{x\to 2}\frac{\sqrt{3x+10}-4}{x-2} = \lim_{x\to 2}\frac{\sqrt{3x+10}-4}{x-2}\cdot\frac{\sqrt{3x+10}+4}{\sqrt{3x+10}+4},$$

$$\lim_{x\to 2}\frac{\sqrt{3x+10}-4}{x-2} = \lim_{x\to 2}\frac{(3x+10)-16}{(x-2)(\sqrt{3x+10}+4)},$$

$$\lim_{x\to 2}\frac{\sqrt{3x+10}-4}{x-2} = \lim_{x\to 2}\frac{3x-6}{(x-2)(\sqrt{3x+10}+4)},$$

$$\lim_{x\to 2}\frac{\sqrt{3x+10}-4}{x-2} = \lim_{x\to 2}\frac{3(x-2)}{(x-2)(\sqrt{3x+10}+4)},$$

$$\lim_{x\to 2}\frac{\sqrt{3x+10}-4}{x-2} = \lim_{x\to 2}\frac{3}{\sqrt{3x+10}+4},$$

$$\lim_{x\to 2}\frac{\sqrt{3x+10}-4}{x-2} = \frac{3}{\sqrt{3(2)+10}+4},$$

$$\boxed{\lim_{x\to 2}\frac{\sqrt{3x+10}-4}{x-2} = \frac{3}{8}.}$$

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Problem 2 Solution

2. Compute the derivative of each function below. Do not simplify your answer.

(a) $\dfrac{x+2\sqrt{x}}{1-x^3}$

(b) $\cos(\cos(2^x))$

(c) $\tan^{-1}\left(\frac{1}{3}\tan(x)\right)$

Solution:

(a) Use the Quotient Rule.

$$\frac{d}{dx}\frac{x+2\sqrt{x}}{1-x^3} = \frac{(1-x^3)\frac{d}{dx}(x+2\sqrt{x}) - (x+2\sqrt{x})\frac{d}{dx}(1-x^3)}{(1-x^3)^2},$$

$$\boxed{\frac{d}{dx}\frac{x+2\sqrt{x}}{1-x^3} = \frac{(1-x^3)(1+\frac{1}{\sqrt{x}}) - (x+2\sqrt{x})(-3x^2)}{(1-x^3)^2}}$$

(b) Use the Chain Rule twice.

$$\frac{d}{dx}\cos(\cos(2^x)) = -\sin(\cos(2^x))\cdot\frac{d}{dx}\cos(2^x),$$

$$\frac{d}{dx}\cos(\cos(2^x)) = -\sin(\cos(2^x))\cdot(-\sin(2^x))\cdot\frac{d}{dx}2^x,$$

$$\boxed{\frac{d}{dx}\cos(\cos(2^x)) = -\sin(\cos(2^x))\cdot(-\sin(2^x))\cdot(\ln 2)2^x}$$

(c) Use the Chain Rule.

$$\frac{d}{dx}\tan^{-1}\left(\frac{1}{3}\tan(x)\right) = \frac{1}{1+(\frac{1}{3}\tan(x))^2}\cdot\frac{d}{dx}\frac{1}{3}\tan(x),$$

$$\boxed{\frac{d}{dx}\tan^{-1}\left(\frac{1}{3}\tan(x)\right) = \frac{1}{1+(\frac{1}{3}\tan(x))^2}\cdot\frac{1}{3}\sec^2(x).}$$

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Problem 3 Solution

3. Consider the function $f(x) = \dfrac{x^3}{x^2 - 4}$.

(a) What is the domain of this function?

(b) Identify all **vertical** asymptotes of the graph of this function. Write an equation for each one. If the graph has no vertical asymptotes, explain why.

(c) Identify all **horizontal** asymptotes of the graph of this function. Write an equation for each one. If the graph has no horizontal asymptotes, explain why.

(d) Find all critical points of this function and classify each one as a local maximum, local minimum, or neither.

(e) Find all inflection points of this function.

Solution:

(a) The domain of $f(x)$ is all real numbers except $x = \pm 2$.

(b) $\boxed{x = 2}$ and $\boxed{x = -2}$ are vertical asymptotes because at least one of the one-sided limits of $f(x)$ as $x \to 2$ and $x \to -2$ is infinite. In fact,

$$\lim_{x \to 2^-} \frac{x^3}{x^2 - 4} = -\infty,$$
$$\lim_{x \to 2^+} \frac{x^3}{x^2 - 4} = +\infty,$$
$$\lim_{x \to -2^-} \frac{x^3}{x^2 - 4} = -\infty,$$
$$\lim_{x \to -2^+} \frac{x^3}{x^2 - 4} = +\infty.$$

(c) There are **no horizontal asymptotes** because the limit of $f(x)$ as $x \to +\infty$ and $x \to -\infty$ are infinite. In fact,

$$\lim_{x \to +\infty} \frac{x^3}{x^2 - 4} = +\infty, \qquad \lim_{x \to -\infty} \frac{x^3}{x^2 - 4} = -\infty.$$

(d) The first derivative of $f(x)$ is

$$f'(x) = \frac{(x^2-4)\frac{d}{dx}x^3 - x^3\frac{d}{dx}(x^2-4)}{(x^2-4)^2},$$
$$f'(x) = \frac{(x^2-4)(3x^2)-(x^3)(2x)}{(x^2-4)^2},$$
$$f'(x) = \frac{3x^4-12x^2-2x^4}{(x^2-4)^2},$$
$$f'(x) = \frac{x^4-12x^2}{(x^2-4)^2}.$$

We know that a critical point of $f(x)$ is a number c that lies in the domain of $f(x)$ and that either $f'(c) = 0$ or $f'(c)$ does not exist. In this case, $f'(x)$ will not exist at $x = \pm 2$. However, neither of these numbers is in the domain of $f(x)$. Therefore, the only critical points will be solutions to $f'(x) = 0$.

$$f'(x) = 0,$$
$$\frac{x^4-12x^2}{(x^2-4)^2} = 0,$$
$$x^4 - 12x^2 = 0,$$
$$x^2(x^2-12) = 0.$$

Either $x^2 = 0$, which gives us $x = 0$, or $x^2 - 12$ which gives us $x = \pm\sqrt{12} = \pm 2\sqrt{3}$. We now use the First Derivative Test to classify the critical points.

Interval	**Test Point,** c	$f'(c)$	**Conclusion**
$(-\infty, -2\sqrt{3})$	-4	$f'(-4) = \frac{64}{144}$	increasing
$(-2\sqrt{3}, -2)$	-3	$f'(-3) = -\frac{27}{25}$	decreasing
$(-2, 0)$	-1	$f'(-1) = -\frac{11}{9}$	decreasing
$(0, 2)$	1	$f'(1) = -\frac{11}{9}$	decreasing
$(2, 2\sqrt{3})$	3	$f'(3) = -\frac{27}{25}$	decreasing
$(2\sqrt{3}, +\infty)$	4	$f'(4) = \frac{64}{144}$	increasing

The first derivative changes sign from positive to negative across $x = -2\sqrt{3}$. Therefore, $x = -2\sqrt{3}$ corresponds to a local maximum of $f(x)$. The first derivative changes sign from negative to positive across $x = 2\sqrt{3}$. Therefore, $x = 2\sqrt{3}$ corresponds to a local minimum of $f(x)$. On the other hand, the first derivative does not change sign across $x = 0$. Therefore, $x = 0$ corresponds to neither a local maximum nor a local minimum.

(e) The second derivative of $f(x)$ is

$$f''(x) = \frac{(x^2-4)^2\frac{d}{dx}(x^4-12x^2) - (x^4-12x^2)\frac{d}{dx}(x^2-4)^2}{(x^2-4)^4},$$

$$f''(x) = \frac{(x^2-4)^2(4x^3-24x) - (x^4-12x^2)\cdot 2(x^2-4)\cdot 2x}{(x^2-4)^4},$$

$$f''(x) = \frac{(x^2-4)[(x^2-4)(4x^3-24x) - 4x(x^4-12x^2)}{(x^2-4)^4},$$

$$f''(x) = \frac{4x^5-40x^3+96x-4x^5+48x^3}{(x^2-4)^3},$$

$$f''(x) = \frac{8x^3}{(x^2-4)^3}.$$

An inflection point of $f(x)$ is a number c in the domain of $f(x)$ such that either $f''(c) = 0$ or $f''(c)$ does not exist and $f''(x)$ changes sign across c. Although $f''(x)$ does not exist at $x = \pm 2$, neither is in the domain of $f(x)$. Therefore, the only possible critical points are solutions to $f''(x) = 0$.

$$\begin{aligned} f''(x) &= 0, \\ \frac{8x^3}{(x^2-4)^3} &= 0, \\ 8x^3 &= 0, \\ x &= 0. \end{aligned}$$

We note that $f''(-1) = \frac{8}{27}$ and $f''(1) = -\frac{8}{27}$. Therefore, since there is a sign change across $x = 0$ we know that $x = 0$ is an inflection point.

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Problem 4 Solution

4. Use a linear approximation to estimate each quantity. Clearly indicate the function and the point where you are taking the linear approximation.

(a) $\sqrt{79}$

(b) $\ln(1.067)$

Solution:

(a) Let $f(x) = \sqrt{x}$ and $a = 81$. The derivative of $f(x)$ is $f'(x) = \frac{1}{2\sqrt{x}}$. Therefore, the linearization of $f(x)$ at $a = 81$ is

$$\begin{aligned} L(x) &= f(81) + f'(81)(x - 81), \\ L(x) &= \sqrt{81} + \frac{1}{2\sqrt{81}}(x - 81), \\ L(x) &= 9 + \frac{1}{18}(x - 1). \end{aligned}$$

An approximate value of $\sqrt{79}$ is $L(79)$. That is,

$$\begin{aligned} \sqrt{79} &\approx L(79), \\ \sqrt{79} &\approx 9 + \frac{1}{18}(79 - 81), \\ \sqrt{79} &\approx 9 - \frac{1}{9}, \end{aligned}$$

$$\boxed{\sqrt{79} \approx \frac{80}{9}.}$$

(b) Let $f(x) = \ln(x)$ and $a = 1$. The derivative of $f(x)$ is $f'(x) = \frac{1}{x}$. Therefore, the linearization of $f(x)$ at $a = 1$ is

$$\begin{aligned} L(x) &= f(1) + f'(1)(x - 1), \\ L(x) &= \ln(1) + \frac{1}{1}(x - 1), \\ L(x) &= x - 1. \end{aligned}$$

An approximate value of $\ln(1.067)$ is $L(1.067)$. That is,

$$\begin{aligned} \ln(1.067) &\approx L(1.067), \\ \ln(1.067) &\approx 1.067 - 1, \end{aligned}$$

$$\boxed{\ln(1.067) \approx 0.067.}$$

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Problem 5 Solution

5. Compute each limit or explain why it does not exist.

(a) $\lim_{x\to 0} \frac{1-\cos(3x)}{x^2}$

(b) $\lim_{x\to 0^+} (3x)^{5x}$

Solution:

(a) This limit is of the form $\frac{0}{0}$ which is indeterminate. We will use L'Hopital's Rule to evaluate the limit.

$$\lim_{x\to 0} \frac{1-\cos(3x)}{x^2} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{\frac{d}{dx}(1-\cos(3x))}{\frac{d}{dx}\,x^2},$$

$$\lim_{x\to 0} \frac{1-\cos(3x)}{x^2} = \lim_{x\to 0} \frac{3\sin(3x)}{2x},$$

$$\lim_{x\to 0} \frac{1-\cos(3x)}{x^2} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{\frac{d}{dx}3\sin(3x)}{\frac{d}{dx}2x},$$

$$\lim_{x\to 0} \frac{1-\cos(3x)}{x^2} = \lim_{x\to 0} \frac{9\cos(3x)}{2},$$

$$\lim_{x\to 0} \frac{1-\cos(3x)}{x^2} = \frac{9\cos(3\cdot 0)}{2},$$

$$\boxed{\lim_{x\to 0} \frac{1-\cos(3x)}{x^2} = \frac{9}{2}.}$$

(b) This limit is of the form 0^0 which is indeterminate. For this type of indeterminate form, we rewrite the limit as

$$\lim_{x\to 0^+} (3x)^{5x} = \lim_{x\to 0^+} \exp\left(\ln(3x)^{5x}\right) = \lim_{x\to 0^+} \exp\left((5x)\ln(3x)\right) = \exp\left(\lim_{x\to 0^+} (5x)\ln(3x)\right).$$

where we note that $\exp(x) = e^x$. The limit in parentheses is of the form $0\cdot -\infty$. However, we can turn it into a limit of the form $-\frac{\infty}{\infty}$ by rewriting the function as

$$(5x)\ln(3x) = \frac{\ln(3x)}{\frac{1}{5x}}.$$

We can then use L'Hopital's Rule.

$$\begin{aligned}
\lim_{x\to 0^+} (5x)\ln(3x) &= \lim_{x\to 0^+} \frac{\ln(3x)}{\frac{1}{5x}}, \\
\lim_{x\to 0^+} (5x)\ln(3x) &\overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{\frac{d}{dx}\ln(3x)}{\frac{d}{dx}\frac{1}{5x}}, \\
\lim_{x\to 0^+} (5x)\ln(3x) &= \lim_{x\to 0^+} \frac{\frac{1}{3x}\cdot 3}{-\frac{1}{5x^2}}, \\
\lim_{x\to 0^+} (5x)\ln(3x) &= \lim_{x\to 0^+} (-5x), \\
\lim_{x\to 0^+} (5x)\ln(3x) &= 0.
\end{aligned}$$

Therefore, the value of the limit in the original problem is

$$\boxed{\lim_{x\to 0^+} (3x)^{5x} = \exp\left(\lim_{x\to 0^+} (5x)\ln(3x)\right) = \exp(0) = 1.}$$

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Problem 6 Solution

6. Suppose that functions $g(x)$ and $h(x)$ satisfy

$$\int_1^5 g(x)\,dx = -4, \quad \int_1^5 h(x)\,dx = 0,$$
$$\int_3^5 g(x)\,dx = -1, \quad \int_3^5 (g(x) - h(x))\,dx = 0,$$

and that $g(x) < 0$ for all x. Calculate each of the following integrals:

(a) $\displaystyle\int_5^1 (g(x) + 1)\,dx$

(b) $\displaystyle\int_1^3 h(x)\,dx$

(c) $\displaystyle\int_1^5 (|g(x)| + 3h(x))\,dx$

Solution:

(a) Using one of the linearity rules, the rule for switching the limits of integration, and the Fundamental Theorem of Calculus, the value of the integral is found to be:

$$\int_5^1 (g(x) + 1)\,dx = \int_5^1 g(x)\,dx + \int_5^1 1\,dx,$$
$$\int_5^1 (g(x) + 1)\,dx = -\int_1^5 g(x)\,dx + \int_5^1 1\,dx,$$
$$\int_5^1 (g(x) + 1)\,dx = -(-4) + \Big[x\Big]_5^1,$$
$$\int_5^1 (g(x) + 1)\,dx = 4 + \Big[1 - 5\Big],$$
$$\int_5^1 (g(x) + 1)\,dx = 0.$$

(b) We begin by noting that, since $\displaystyle\int_3^5 (g(x) - h(x))\,dx$, we know that

$$\int_3^5 h(x)\,dx = \int_3^5 g(x)\,dx = -1.$$

Furthermore, the property that allows us to split an integral into two integrals gives us the equation

$$\int_1^5 h(x)\,dx = \int_1^3 h(x)\,dx + \int_3^5 h(x)\,dx.$$

Therefore, we have

$$\begin{aligned}\int_1^5 h(x)\,dx &= \int_1^3 h(x)\,dx + \int_3^5 h(x)\,dx,\\ 0 &= \int_1^3 h(x)\,dx - 1,\\ 1 &= \int_1^3 h(x)\,dx.\end{aligned}$$

(c) Using the linearity rules for definite integrals we can rewrite the given integral as follows:

$$\int_1^5 (|g(x)| + 3h(x))\,dx = \int_1^5 |g(x)|\,dx + 3\int_1^5 h(x)\,dx.$$

Using the fact that $g(x) < 0$ for all x we can say that

$$\int_1^5 |g(x)|\,dx = \int_1^5 (-g(x))\,dx = -\int_1^5 g(x)\,dx = -(-4) = 4.$$

Furthermore, since $\displaystyle\int_1^5 h(x)\,dx = 0$ the value of the integral is

$$\int_1^5 (|g(x)| + 3h(x))\,dx = \int_1^5 |g(x)|\,dx + 3\int_1^5 h(x)\,dx = 4 + 3(0) = 4.$$

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Problem 7 Solution

7. Compute the definite integrals.

(a) $\displaystyle\int_{-7\pi/16}^{7\pi/16} (1 + \tan^3(x))\,dx$

(b) $\displaystyle\int_1^2 \frac{2 + 3x}{\sqrt{x}}\,dx$

Solution:

(a) We begin by splitting the integral into the sum of two integrals:

$$\int_{-7\pi/16}^{7\pi/16} (1 + \tan^3(x))\,dx = \int_{-7\pi/16}^{7\pi/16} 1\,dx + \int_{-7\pi/16}^{7\pi/16} \tan^3(x)\,dx.$$

We note that 1 is an even function so that

$$\int_{-7\pi/16}^{7\pi/16} 1\,dx = 2\int_0^{7\pi/16} 1\,dx = 2\Big[x\Big]_0^{7\pi/16} = \frac{7\pi}{8},$$

and that $\tan^3(x)$ is an odd function so that

$$\int_{-7\pi/16}^{7\pi/16} \tan^3(x)\,dx = 0.$$

Therefore,

$$\int_{-7\pi/16}^{7\pi/16} (1 + \tan^3(x))\,dx = \int_{-7\pi/16}^{7\pi/16} 1\,dx + \int_{-7\pi/16}^{7\pi/16} \tan^3(x)\,dx = \boxed{\frac{7\pi}{8}}.$$

(b) We solve the integral by rewriting the integrand and using the Fundamental Theorem of Calculus.

$$\int_1^2 \frac{2+3x}{\sqrt{x}}\,dx = \int_1^2 \left(\frac{2}{\sqrt{x}} + \frac{3x}{\sqrt{x}}\right)\,dx,$$

$$\int_1^2 \frac{2+3x}{\sqrt{x}}\,dx = \int_1^2 \left(2x^{-1/2} + 3x^{1/2}\right)\,dx,$$

$$\int_1^2 \frac{2+3x}{\sqrt{x}}\,dx = 2\int_1^2 x^{-1/2}\,dx + 3\int_1^2 x^{1/2}\,dx,$$

$$\int_1^2 \frac{2+3x}{\sqrt{x}}\,dx = 2\Big[2x^{1/2}\Big]_1^2 + 3\left[\frac{2}{3}x^{3/2}\right]_1^2,$$

$$\int_1^2 \frac{2+3x}{\sqrt{x}}\,dx = 2\Big[2\sqrt{2} - 2\Big] + 3\left[\frac{2}{3}(2)^{3/2} - \frac{2}{3}\right],$$

$$\int_1^2 \frac{2+3x}{\sqrt{x}}\,dx = 4\sqrt{2} - 4 + 4\sqrt{2} - 2,$$

$$\boxed{\int_1^2 \frac{2+3x}{\sqrt{x}}\,dx = 8\sqrt{2} - 6.}$$

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Problem 8 Solution

8. Compute the indefinite integrals.

(a) $\int \cos^2(x)\,dx$

(b) $\int \frac{3x}{\sqrt{x^2+7}}\,dx$

Solution:

(a) To solve this integral we must use the double angle identity

$$\cos^2(x) = \frac{1+\cos(2x)}{2}.$$

Making this replacement and evaluating the integral we find that

$$\int \cos^2(x)\,dx = \int \frac{1+\cos(2x)}{2}\,dx,$$

$$\int \cos^2(x)\,dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right)\,dx,$$

$$\boxed{\int \cos^2(x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C.}$$

(b) We use the u-substitution to evaluate the integral. Let $u = x^2 + 7$. Then $\frac{1}{2}\,du = x\,dx$. Making the substitutions and evaluating we get

$$\int \frac{3x}{\sqrt{x^2+7}}\,dx = 3\int \frac{1}{\sqrt{x^2+7}} \cdot x\,dx,$$

$$\int \frac{3x}{\sqrt{x^2+7}}\,dx = 3\int \frac{1}{\sqrt{u}} \cdot \frac{1}{2}\,du,$$

$$\int \frac{3x}{\sqrt{x^2+7}}\,dx = \frac{3}{2}\int u^{-1/2}\,du,$$

$$\int \frac{3x}{\sqrt{x^2+7}}\,dx = \frac{3}{2} \cdot 2\sqrt{u} + C,$$

$$\boxed{\int \frac{3x}{\sqrt{x^2+7}}\,dx = 3\sqrt{x^2+7} + C.}$$