Math 181, Exam 1, Fall 2008
Problem 1 Solution

1. Compute the following integrals.

(a) $\displaystyle\int \frac{\sin x}{1 - 2\cos x}\,dx$

(b) $\displaystyle\int \frac{dx}{\sqrt{1 - 4x^2}}$

Solution:

(a) The integral is computed using the u-substitution method. Let $u = 1 - 2\cos x$. Then $du = 2\sin x\,dx \quad\Rightarrow\quad \frac{1}{2}\,du = \sin x\,dx$. Substituting these into the integral and evaluating we get:

$$\begin{aligned}
\int \frac{\sin x}{1 - 2\cos x}\,dx &= \int \frac{1}{1 - 2\cos x} \cdot \sin x\,dx \\
&= \int \frac{1}{u} \cdot \frac{1}{2}\,du \\
&= \frac{1}{2}\int \frac{1}{u}\,du \\
&= \frac{1}{2}\ln|u| + C \\
&= \boxed{\frac{1}{2}\ln|1 - 2\cos x| + C}
\end{aligned}$$

(b) The integral is computed using the u-substitution method. Let $u = 2x$. Then $du = 2\,dx \quad\Rightarrow\quad \frac{1}{2}\,du = dx$ and we get:

$$\begin{aligned}
\int \frac{dx}{\sqrt{1 - 4x^2}} &= \int \frac{dx}{\sqrt{1 - (2x)^2}} \\
&= \int \frac{\frac{1}{2}\,du}{\sqrt{1 - u^2}} \\
&= \frac{1}{2}\int \frac{1}{\sqrt{1 - u^2}}\,du \\
&= \frac{1}{2}\arcsin u + C \\
&= \boxed{\frac{1}{2}\arcsin(2x) + C}
\end{aligned}$$

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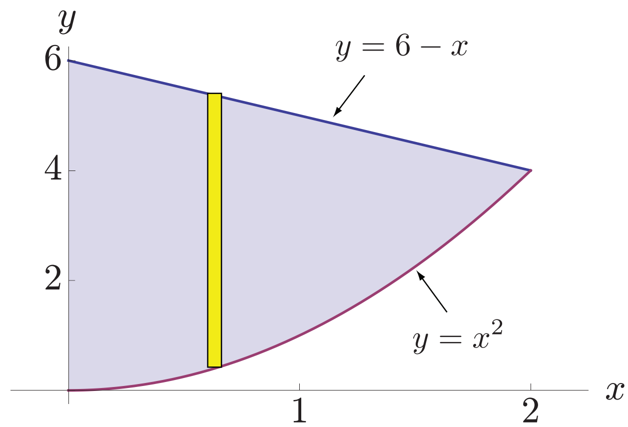
Problem 2 Solution

2. Find the volume of the solid of revolution obtained by rotating the region in the first quadrant bounded by $y = x^2$, $x + y = 6$, and $x = 0$ about the y-axis.

Solution:



To find the volume we will use the **Shell Method**. The variable of integration is x and the formula is:

$$V = 2\pi \int_a^b x\,(\text{top} - \text{bottom})\,dx$$

where the top curve is $y = 6 - x$ and the bottom curve is $y = x^2$. The lower limit of integration is $a = 0$. To determine the upper limit we must find the points of intersection of the top and bottom curves. To do this we set the y's equal to each other and solve for x.

$$\begin{aligned} y &= y \\ x^2 &= 6 - x \\ x^2 + x - 6 &= 0 \\ (x+3)(x-2) &= 0 \\ x = -3,\ x &= 2 \end{aligned}$$

In the problem statement we are told to take the region in the first quadrant. Therefore, we take $b = 2$. The volume is then:

$$\begin{aligned}
V &= 2\pi \int_0^2 x\left[(6-x) - x^2\right]\,dx \\
&= 2\pi \int_0^2 \left(6x - x^2 - x^3\right)\,dx \\
&= 2\pi \left[3x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^2 \\
&= 2\pi \left[3(2)^2 - \frac{1}{3}(2)^3 - \frac{1}{4}(2)^4\right] \\
&= \boxed{\frac{32\pi}{3}}
\end{aligned}$$

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Problem 3 Solution

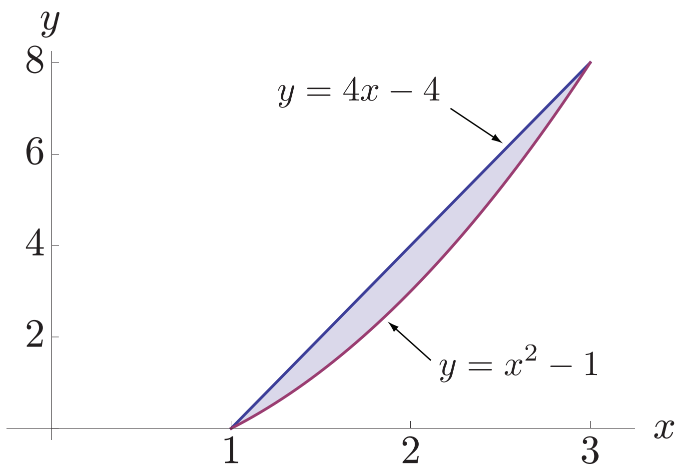
3. Compute each part below.

(a) Compute the area of the region bounded by $y = x^2 - 1$ and $y = 4x - 4$.

(b) Compute $f'(x)$ where

$$f(x) = \int_1^{x^2} \ln(t)\,dt, \qquad x > 0.$$

Solution:



(a)

The formula we will use to compute the area of the region is:

$$\text{Area} = \int_a^b (\text{top} - \text{bottom})\,dx$$

where the limits of integration are the x-coordinates of the points of intersection of the two curves. These are found by setting the y's equal to each other and solving for x.

$$\begin{aligned} y &= y \\ x^2 - 1 &= 4x - 4 \\ x^2 - 4x + 3 &= 0 \\ (x-1)(x-3) &= 0 \\ x = 1,\ x &= 3 \end{aligned}$$

From the graph we see that the top curve is $y = 4x - 4$ and the bottom curve is $y = x^2 - 1$. Therefore, the area between the curves is:

$$\begin{aligned}
\text{Area} &= \int_a^b (\text{top} - \text{bottom})\ dx \\
&= \int_1^3 \left[(4x-4) - (x^2-1)\right]\ dx \\
&= \int_1^3 \left(-x^2 + 4x - 3\right)\ dx \\
&= \left[-\frac{1}{3}x^3 + 2x^2 - 3x\right]_1^3 \\
&= \left[-\frac{1}{3}(3)^3 + 2(3)^2 - 3(3)\right] - \left[-\frac{1}{3}(1)^3 + 2(1)^2 - 3(1)\right] \\
&= [-9 + 18 - 9] - \left[-\frac{1}{3} + 2 - 3\right] \\
&= \boxed{\frac{4}{3}}
\end{aligned}$$

(b) Using the Fundamental Theorem of Calculus Part II and the Chain Rule, the derivative is:

$$\begin{aligned}
F'(x) &= \frac{d}{dx}\int_1^{x^2} \ln(t)\,dt \\
&= \ln\left(x^2\right)\cdot\frac{d}{dx}\left(x^2\right) \\
&= \boxed{\ln\left(x^2\right)\cdot(2x)}
\end{aligned}$$

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Problem 4 Solution

4. Use an integral to compute the volume of a right circular cone whose base has radius R and whose height is h.

Solution: To find the volume we will use the formula:

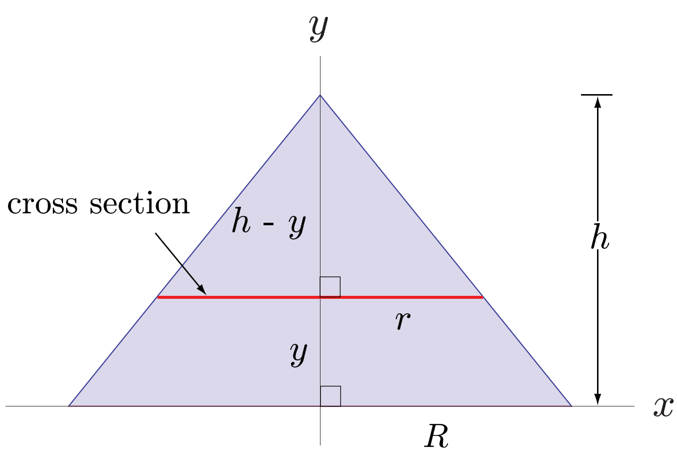
$$V = \int_c^d A(y)\,dy$$

where $A(y)$ is the cross-sectional area of the cone as a function of height y and $0 \le y \le h$. The horizontal cross sections are circles, so the cross-sectional area is:

$$A(y) = \pi r^2$$

where r is the radius of the cross section at height y from the base. If we look at the cone from the side, we see a triangle. The cross-section as viewed from the side is a horizontal line segment at height y. The radius of the cross section is half of the length of this line segment. Using similar triangles, we have:

$$\frac{\text{base}}{\text{height}} = \frac{R}{h} = \frac{r}{h-y}$$
$$r = \frac{R}{h}(h-y)$$



The volume is then:

$$
\begin{aligned}
V &= \int_0^h \pi r^2 \, dy \\
&= \int_0^h \pi \left[\frac{R}{h}(h-y) \right]^2 \, dy \\
&= \pi \frac{R^2}{h^2} \int_0^h (y-h)^2 \, dy \\
&= \pi \frac{R^2}{h^2} \left[\frac{1}{3}(y-h)^3 \right]_0^h \\
&= \pi \frac{R^2}{h^2} \left[\frac{1}{3}(h-h)^3 - \frac{1}{3}(0-h)^3 \right] \\
&= \boxed{\frac{1}{3}\pi R^2 h}
\end{aligned}
$$

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Problem 5 Solution

5. Approximate the value of the definite integral:

$$\int_1^3 \frac{dx}{x}$$

using

(a) the Midpoint Rule with $N = 2$,

(b) the Trapezoidal Rule with $N = 2$, and

(c) Simpson's Rule with $N = 4$.

Your answers should be writen as a single, reduced fraction.

Solution:

(a) The length of each subinterval of $[1, 3]$ is

$$\Delta x = \frac{b-a}{N} = \frac{3-1}{2} = 1$$

The estimate M_2 is:

$$\begin{aligned} M_2 &= \Delta x \left[f\left(\frac{3}{2}\right) + f\left(\frac{5}{2}\right) \right] \\ &= 1 \cdot \left[\frac{1}{\frac{3}{2}} + \frac{1}{\frac{5}{2}} \right] \\ &= \frac{2}{3} + \frac{2}{5} \\ &= \boxed{\frac{16}{15}} \end{aligned}$$

(b) The length of each subinterval of $[1, 3]$ is $\Delta x = 1$ just as in part (a). The estimate T_2 is:

$$\begin{aligned} T_2 &= \frac{\Delta x}{2} [f(1) + 2f(2) + f(3)] \\ &= \frac{1}{2} \left[\frac{1}{1} + 2 \cdot \frac{1}{2} + \frac{1}{3} \right] \\ &= \boxed{\frac{7}{6}} \end{aligned}$$

(c) We can use the following formula to find S_4:

$$S_4 = \frac{2}{3}M_2 + \frac{1}{3}T_2$$

where M_2 and T_2 were found in parts (a) and (b). We get:

$$\begin{aligned} S_4 &= \frac{2}{3}\left(\frac{16}{15}\right) + \frac{1}{3}\left(\frac{7}{6}\right) \\ &= \frac{32}{45} + \frac{7}{18} \\ &= \boxed{\frac{11}{10}} \end{aligned}$$