Math 181, Exam 1, Fall 2009
Problem 1 Solution

1. **a)** i) Find an antiderivative for the function $f(x) = x \cos x$.

ii) Compute the definite integral $\displaystyle\int_0^{\pi} x \cos x \, dx$.

b) i) Find an antiderivative for the function $f(x) = xe^x$.

ii) Compute the definite integral $\displaystyle\int_0^{1} xe^x \, dx$.

Solution: **a)** i) An antiderivative for f is a function F such that:

$$F(x) = \int f(x)\,dx = \int x \cos x \, dx$$

We use Integration by Parts to evaluate the integral. Let $u = x$ and $v' = \cos x$. Then $u' = 1$ and $v = \sin x$. Using the Integration by Parts formula:

$$\int uv' \, dx = uv - \int u'v \, dx$$

we get:

$$F(x) = \int x \cos x \, dx = x \sin x - \int \sin x \, dx$$
$$F(x) = \int x \cos x \, dx = \boxed{x \sin x + \cos x}$$

ii) The value of the integral is found using the Fundamental Theorem of Calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x) = x \sin x + \cos x$ was found in part i).

$$\begin{aligned}\int_0^{\pi} x \cos x \, dx &= F(\pi) - F(0) \\ &= (\pi \sin \pi + \cos \pi) - (0 \cdot \sin 0 + \cos 0) \\ &= (0 - 1) - (0 + 1) \\ &= \boxed{-2}\end{aligned}$$

b) i) An antiderivative for f is a function F such that:

$$F(x) = \int f(x)\,dx = \int xe^x \, dx$$

We use Integration by Parts to evaluate the integral. Let $u = x$ and $v' = e^x$. Then $u' = 1$ and $v = e^x$. Using the Integration by Parts formula:

$$\int uv'\,dx = uv - \int u'v\,dx$$

we get:

$$F(x) = \int xe^x\,dx = xe^x - \int e^x\,dx$$
$$F(x) = \int xe^x\,dx = \boxed{xe^x - e^x}$$

ii) The value of the integral is found using the Fundamental Theorem of Calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x) = xe^x - e^x$ was found in part i).

$$\begin{aligned}\int_0^1 xe^x\,dx &= F(1) - F(0)\\ &= (1 \cdot e^1 - e^1) - (0 \cdot e^0 - e^0)\\ &= (e - e) - (0 - 1)\\ &= \boxed{1}\end{aligned}$$

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Problem 2 Solution

2. **a)** i) Find the trapezoidal approximation T_2 for the function $f(x) = x^2 + x$ over the interval $[0, 2]$.
ii) Find the area enclosed between the graphs of the functions $y = x^2$ and $y = 2 - x^2$.
b) i) Find the midpoint approximation M_2 for the function $f(x) = x^2 - x$ over the interval $[0, 4]$.
ii) Find the area enclosed between the graphs of the functions $y = x^2$ and $y = 3x - 2$.

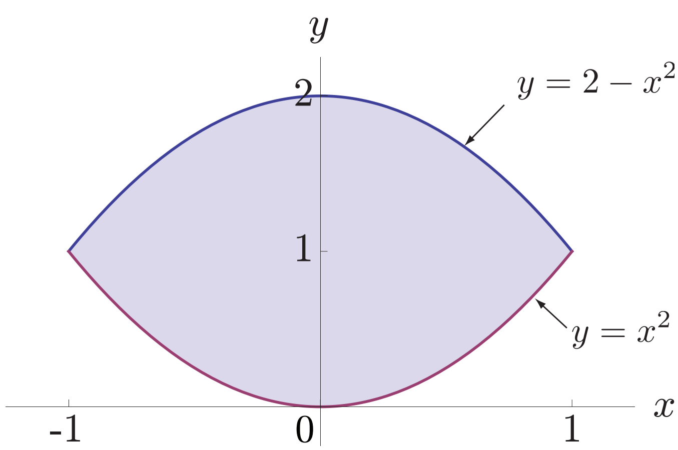
Solution: **a)** i) In this problem we use $N = 2$ subintervals of $[0, 2]$. This is because of the subscript on T_2. The value of Δx is then:

$$\Delta x = \frac{b-a}{N} = \frac{2-0}{2} = 1$$

The trapezoidal approximation T_2 is then:

$$\begin{aligned} T_2 &= \frac{\Delta x}{2}\left[f(0) + 2f(1) + f(2)\right] \\ &= \frac{1}{2}\left[(0^2+0) + 2(1^2+1) + (2^2+2)\right] \\ &= \frac{1}{2}\left[0 + 2(2) + 6\right] \\ &= \boxed{5} \end{aligned}$$

ii)



The formula we will use to compute the area of the region is:

$$\text{Area} = \int_a^b (\text{top} - \text{bottom})\; dx$$

where the limits of integration are the x-coordinates of the points of intersection of the two curves. These are found by setting the y's equal to each other and solving for x.

$$\begin{aligned} y &= y \\ x^2 &= 2 - x^2 \\ 2x^2 &= 2 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

From the figure we see that the top curve is $y = 2 - x^2$ and the bottom one is $y = x^2$. The area is then:

$$\begin{aligned} \text{Area} &= \int_a^b [\text{top} - \text{bottom}]\, dx \\ &= \int_{-1}^{1} \left[\left(2 - x^2\right) - x^2\right]\, dx \\ &= \int_{-1}^{1} \left(2 - 2x^2\right)\, dx \\ &= \left[2x - \frac{2}{3}x^3\right]_{-1}^{1} \\ &= \left[2(1) - \frac{2}{3}(1)^3\right] - \left[2(-1) - \frac{2}{3}(-1)^3\right] \\ &= \left[2 - \frac{2}{3}\right] - \left[-2 + \frac{2}{3}\right] \\ &= \boxed{\frac{8}{3}} \end{aligned}$$

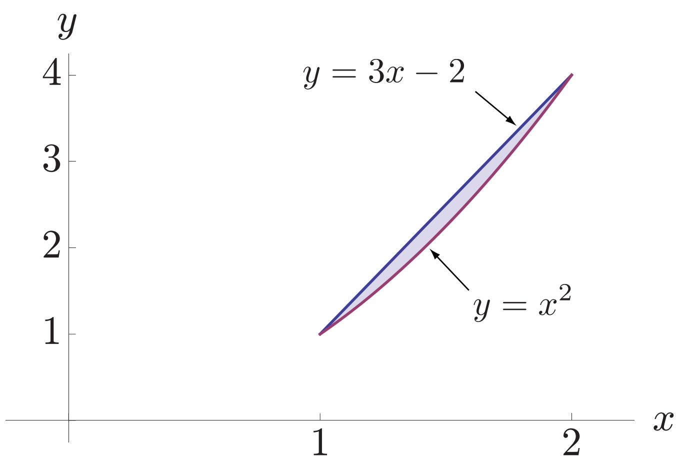
b) i) In this problem we use $N = 2$ subintervals of $[0, 4]$. This is because of the subscript on M_2. The value of Δx is then:

$$\Delta x = \frac{b - a}{N} = \frac{4 - 0}{2} = 2$$

The midpoint approximation M_2 is then:

$$\begin{aligned} M_2 &= \Delta x\, [f(1) + f(3)] \\ &= 2 \cdot \left[(1^2 - 1) + (3^2 - 3)\right] \\ &= 2 \cdot [0 + 6] \\ &= \boxed{12} \end{aligned}$$

ii)



The formula we will use to compute the area of the region is:

$$\text{Area} = \int_a^b (\text{top} - \text{bottom})\ dx$$

where the limits of integration are the x-coordinates of the points of intersection of the two curves. These are found by setting the y's equal to each other and solving for x.

$$\begin{aligned} y &= y \\ x^2 &= 3x - 2 \\ x^2 - 3x + 2 &= 0 \\ (x-1)(x-2) &= 0 \\ x = 1,\ x &= 2 \end{aligned}$$

From the graph we see that the top graph is $y = 3x - 2$ and the bottom one is $y = x^2$. The area is then:

$$\begin{aligned} \text{Area} &= \int_a^b [\text{top} - \text{bottom}]\ dx \\ &= \int_1^2 \left[(3x-2) - x^2\right]\ dx \\ &= \int_1^2 \left(3x - 2 - x^2\right)\ dx \\ &= \left[\frac{3}{2}x^2 - 2x - \frac{1}{3}x^3\right]_1^2 \\ &= \left[\frac{3}{2}(2)^2 - 2(2) - \frac{1}{3}(2)^3\right] - \left[\frac{3}{2}(1)^2 - 2(1) - \frac{1}{3}(1)^3\right] \\ &= \left[6 - 4 - \frac{8}{3}\right] - \left[\frac{3}{2} - 2 - \frac{1}{3}\right] \\ &= \boxed{\frac{1}{6}} \end{aligned}$$

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Problem 3 Solution

3. **a)** Compute the indefinite integrals:

$$\int \frac{dx}{x^2+x} \qquad \int \sin^3 x\, dx$$

b) Compute the indefinite integrals:

$$\int \frac{dx}{x^2-1} \qquad \int \cos^3 x\, dx$$

Solution: **a)** The first integral can be solved using Partial Fraction Decomposition. First, we factor the denominator and then decompose the rational function into a sum of simpler rational functions.

$$\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

Next, we multiply the above equation by $x(x+1)$ to get:

$$1 = A(x+1) + Bx$$

Then we plug in two different values for x to create a system of two equations in two unknowns (A,B). We select $x=0$ and $x=-1$ for simplicity.

$$x=0: \;\; A(0+1)+B(0)=1 \quad \Rightarrow \quad A=1$$
$$x=-1: \;\; A(-1+1)+B(-1)=1 \quad \Rightarrow \quad B=-1$$

Finally, we plug these values for A and B back into the decomposition and integrate.

$$\int \frac{dx}{x^2+x} = \int \left(\frac{1}{x} + \frac{-1}{x+1}\right) dx$$
$$= \boxed{\ln|x| - \ln|x+1| + C}$$

The second integral can be solved by rewriting it using the Pythagorean Identity $\cos^2 x + \sin^2 x = 1$.

$$\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx$$
$$= \int \left(1-\cos^2 x\right) \sin x\, dx$$

Now let $u = \cos x$. Then $du = -\sin x \, dx \quad \Rightarrow -du = \sin x \, dx$ and we get:

$$\begin{aligned}
\int \sin^3 x \, dx &= \int \left(1 - \cos^2 x\right) \sin x \, dx \\
&= \int \left(1 - u^2\right)(-du) \\
&= \int \left(u^2 - 1\right) du \\
&= \frac{1}{3}u^3 - u + C \\
&= \boxed{\frac{1}{3}\cos^3 x - \cos x + C}
\end{aligned}$$

b) The first integral can be solved using Partial Fraction Decomposition. First, we factor the denominator and then decompose the rational function into a sum of simpler rational functions.

$$\frac{1}{x^2 - 1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

Next, we multiply the above equation by $x(x+1)$ to get:

$$1 = A(x+1) + B(x-1)$$

Then we plug in two different values for x to create a system of two equations in two unknowns (A,B). We select $x = 1$ and $x = -1$ for simplicity.

$$\begin{aligned}
x = 1: \; A(1+1) + B(1-1) = 1 \quad &\Rightarrow \quad A = \frac{1}{2} \\
x = -1: \; A(-1+1) + B(-1-1) = 1 \quad &\Rightarrow \quad B = -\frac{1}{2}
\end{aligned}$$

Finally, we plug these values for A and B back into the decomposition and integrate.

$$\begin{aligned}
\int \frac{dx}{x^2 - 1} &= \int \left(\frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1} \right) dx \\
&= \boxed{\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C}
\end{aligned}$$

The second integral can be solved by rewriting it using the Pythagorean Identity $\cos^2 x + \sin^2 x = 1$.

$$\begin{aligned}
\int \cos^3 x \, dx &= \int \cos^2 x \cos x \, dx \\
&= \int \left(1 - \sin^2 x\right) \cos x \, dx
\end{aligned}$$

Now let $u = \sin x$. Then $du = \cos x\, dx$ and we get:

$$\begin{aligned}
\int \cos^3 x\, dx &= \int \left(1 - \sin^2 x\right) \cos x\, dx \\
&= \int \left(1 - u^2\right) du \\
&= u - \frac{1}{3}u^3 + C \\
&= \boxed{\sin x - \frac{1}{3}\sin^3 x + C}
\end{aligned}$$

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Problem 4 Solution

4.**a)** Compute the indefinite integrals:

$$\int \frac{\ln x \, dx}{x} \qquad \int \arctan x \, dx$$

b) Compute the indefinite integrals:

$$\int \frac{\arctan x}{x^2+1} \, dx \qquad \int \ln x \, dx$$

Solution: **a)** The first integral is computed using the u-substitution method. Let $u = \ln x$. Then $du = \frac{1}{x} \, dx$ and we get:

$$\begin{aligned} \int \frac{\ln x \, dx}{x} &= \int u \, du \\ &= \frac{1}{2} u^2 + C \\ &= \boxed{\frac{1}{2}(\ln x)^2 + C} \end{aligned}$$

The second integral is computed using Integration by Parts. Let $u = \arctan x$ and $v' = 1$. Then $u' = \frac{1}{x^2+1}$ and $v = x$. Using the Integration by Parts formula:

$$\int uv' \, dx = uv - \int u'v \, dx$$

we get:

$$\int \arctan x \, dx = x \arctan x - \int \frac{x}{x^2+1} \, dx$$

The integral on the right hand side is computed using the u-substitution $u = x^2 + 1$. Then

$du = 2x\,dx \quad \Rightarrow \quad \frac{1}{2}\,du = x\,dx$ and we get:

$$\begin{aligned}
\int \arctan x\,dx &= x\arctan x - \int \frac{x}{x^2+1}\,dx \\
&= x\arctan x - \int \frac{1}{x^2+1}\cdot x\,dx \\
&= x\arctan x - \int \frac{1}{u}\cdot\frac{1}{2}\,du \\
&= x\arctan x - \frac{1}{2}\int \frac{1}{u}\,du \\
&= x\arctan x - \frac{1}{2}\ln|u| + C \\
&= \boxed{x\arctan x - \frac{1}{2}\ln(x^2+1) + C}
\end{aligned}$$

b) The first integral is computed using the u-substitution method. Let $u = \arctan x$. Then $du = \dfrac{1}{x^2+1}\,dx$ and we get:

$$\begin{aligned}
\int \frac{\arctan x}{x^2+1}\,dx &= \int u\,du \\
&= \frac{1}{2}u^2 + C \\
&= \boxed{\frac{1}{2}(\arctan x)^2 + C}
\end{aligned}$$

The second integral is computed using Integration by Parts. Let $u = \ln x$ and $v' = 1$. Then $u' = \dfrac{1}{x}$ and $v = x$. Using the Integration by Parts formula:

$$\int uv'\,dx = uv - \int u'v\,dx$$

we get:

$$\begin{aligned}
\int \ln x\,dx &= x\ln x - \int \frac{1}{x}\cdot x\,dx \\
&= x\ln x - \int dx \\
&= \boxed{x\ln x - x + C}
\end{aligned}$$

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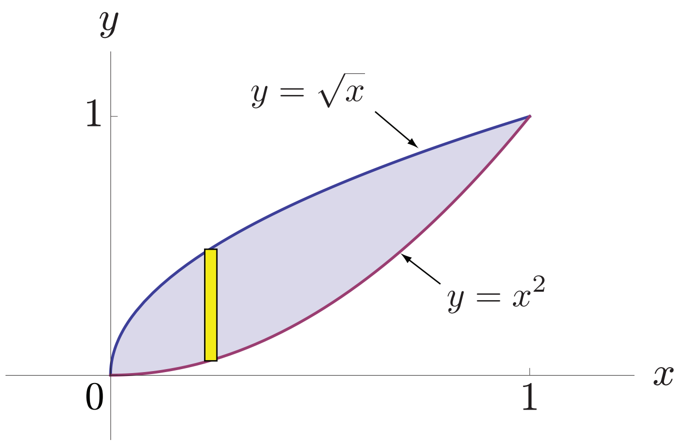
Problem 5 Solution

5.**a)** The region enclosed by the graphs of $y = x^2$ and $y = \sqrt{x}$ is rotated about the x-axis. Find the volume of the resulting solid.
b) The region enclosed by the graphs of $y = x^2$ and $y = \sqrt{x}$ is rotated about the y-axis. Find the volume of the resulting solid.

Solution:



a) To find the volume of the solid obtained when the region is rotated about the x-axis, we will use the **Washer Method**. The variable of integration is x and the corresponding formula is:

$$V = \pi \int_a^b \left[(\text{top})^2 - (\text{bottom})^2 \right] \, dx$$

The top curve is $y = \sqrt{x}$ and the bottom curve is $y = x^2$. The values of a and b correspond to the points of intersection of the two graphs. To determine these we set the y's equal to each other and solve for x.

$$\begin{aligned} y &= y \\ x^2 &= \sqrt{x} \\ x^4 &= x \\ x^4 - x &= 0 \\ x(x^3 - 1) &= 0 \\ x = 0,\ x^3 &= 1 \quad \Rightarrow \quad x = 1 \end{aligned}$$

Therefore, the volume is:

$$\begin{aligned} V &= \pi \int_0^1 \left[\left(\sqrt{x}\right)^2 - \left(x^2\right)^2 \right] dx \\ &= \pi \int_0^1 \left(x - x^4\right) dx \\ &= \pi \left[\frac{1}{2}x^2 - \frac{1}{5}x^5 \right]_0^1 \\ &= \pi \left[\frac{1}{2} - \frac{1}{5} \right] \\ &= \boxed{\frac{3\pi}{10}} \end{aligned}$$

b) To find the volume of the solid obtained when the region is rotated about the y-axis, we will use the **Shell Method**. The variable of integration is x and the corresponding formula is:

$$V = 2\pi \int_a^b x\,(\text{top} - \text{bottom})\; dx$$

The top curve is $y = \sqrt{x}$ and the bottom curve is $y = x^2$. The values of a and b are $a = 0$ and $b = 1$ as found in part (a). Therefore, the volume is:

$$\begin{aligned} V &= 2\pi \int_0^1 x\left(\sqrt{x} - x^2\right) dx \\ &= 2\pi \int_0^1 \left(x^{3/2} - x^3\right) dx \\ &= 2\pi \left[\frac{2}{5}x^{5/2} - \frac{1}{4}x^4 \right]_0^1 \\ &= 2\pi \left[\frac{2}{5} - \frac{1}{4} \right] \\ &= \boxed{\frac{3\pi}{10}} \end{aligned}$$