Math 181, Exam 1, Fall 2011
Problem 1 Solution

1. Compute the indefinite integral $\int \cos^7 x\, dx$.

Solution: The integral can be solved by rewriting it using the Pythagorean Identity $\cos^2 x + \sin^2 x = 1$.

$$\begin{aligned}
\int \cos^7 x\, dx &= \int \cos^6 x \cos x\, dx \\
&= \int (\cos^2 x)^3 \cos\ dx \\
&= \int (1 - \sin^2 x)^3 \cos x\, dx
\end{aligned}$$

Now let $u = \sin x$. Then $du = \cos x\, dx$ and we get:

$$\begin{aligned}
\int \cos^7 x\, dx &= \int \left(1 - \sin^2 x\right)^3 \cos x\, dx \\
&= \int \left(1 - u^2\right)^3\, du \\
&= \int \left(1 - 3u^2 + 3u^4 - u^6\right)\, du \\
&= u - u^3 + \frac{3}{5}u^5 - \frac{1}{7}u^7 + C \\
&= \boxed{\sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C}
\end{aligned}$$

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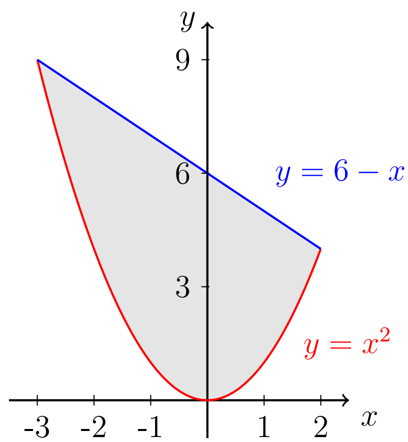
Problem 2 Solution

2. Find the volume of the solid obtained by rotating about the x-axis the region enclosed by the graphs of $y = x^2$ and $y = 6 - x$.

Solution: The region being rotated about the x-axis is shown below.



We find the volume using the Washer method. The formula we will use is:

$$V = \pi \int_a^b \left(\text{top}^2 - \text{bottom}^2\right)\, dx$$

where the top curve is $y = 6 - x$ and the bottom curve is $y = x^2$. The limits of integration are the x-coordinates of the points of intersection of the two graphs. To find the limits of integration, we set the y's equal to each other and solve for x.

$$\begin{aligned} y &= y \\ x^2 &= 6 - x \\ x^2 + x - 6 &= 0 \\ (x+3)(x-2) &= 0 \\ x = -3,\ x &= 2 \end{aligned}$$

The volume is then:

$$
\begin{aligned}
V &= \pi \int_a^b \left(\text{top}^2 - \text{bottom}^2\right)\, dx \\
&= \pi \int_{-3}^{2} \left[(6-x)^2 - \left(x^2\right)^2\right]\, dx \\
&= \pi \int_{-3}^{2} \left(36 - 12x + x^2 - x^4\right)\, dx \\
&= \pi \left[36x - 6x^2 + \frac{1}{3}x^3 - \frac{1}{5}x^5\right]_{-3}^{2} \\
&= \pi \left[\left(36(2) - 6(2)^2 + \frac{1}{3}(2)^3 - \frac{1}{5}(2)^5\right) - \left(36(-3) - 6(-3)^2 + \frac{1}{3}(-3)^3 - \frac{1}{5}(-3)^5\right)\right] \\
&= \boxed{\frac{500\pi}{3}}
\end{aligned}
$$

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Problem 3 Solution

3. Compute the indefinite integral:

$$\int x^3 \ln x \, dx$$

Solution: We will evaluate the integral using Integration by Parts. Let $u = \ln x$ and $v' = x^3$. Then $u' = \frac{1}{x}$ and $v = \frac{1}{4}x^4$. Using the Integration by Parts formula:

$$\int uv' \, dx = uv - \int u'v \, dx$$

we get:

$$\begin{aligned}\int x^3 \ln x \, dx &= \frac{1}{4}x^4 \ln x - \int \left(\frac{1}{x}\right)\left(\frac{1}{4}x^4\right) dx \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 \, dx \\ &= \boxed{\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C}\end{aligned}$$

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Problem 4 Solution

4. Compute the indefinite integral:

$$\int \frac{dx}{x^2 - 3x + 2}$$

Solution: We will evaluate the integral using Partial Fraction Decomposition. First, we factor the denominator and then decompose the rational function into a sum of simpler rational functions.

$$\frac{1}{x^2 + 3x + 2} = \frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

Next, we multiply the above equation by $(x+1)(x+2)$ to get:

$$1 = A(x+2) + B(x+1)$$

Then we plug in two different values for x to create a system of two equations in two unknowns (A,B). We select $x = -1$ and $x = -2$ for simplicity.

$$\begin{aligned} x = -1: &\quad A(-1+2) + B(-1+1) = 1 \quad \Rightarrow \quad A = 1 \\ x = -2: &\quad A(-2+2) + B(-2+1) = 1 \quad \Rightarrow \quad B = -1 \end{aligned}$$

Finally, we plug these values for A and B back into the decomposition and integrate.

$$\begin{aligned} \int \frac{dx}{x^2 + 3x + 2} &= \int \left(\frac{A}{x+1} + \frac{B}{x+2} \right) dx \\ &= \int \left(\frac{1}{x+1} + \frac{-1}{x+2} \right) dx \\ &= \boxed{\ln|x+1| - \ln|x+2| + C} \end{aligned}$$

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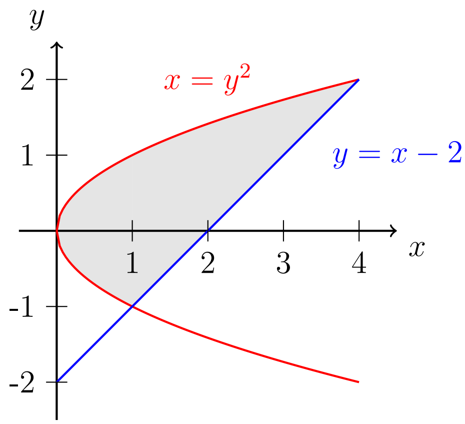
Problem 5 Solution

5. Find the area of the region enclosed by the curves $y^2 = x$ and $y = x - 2$.

Solution:



The formula we will use to compute the area of the region is:

$$\text{Area} = \int_c^d (\text{right} - \text{left})\, dx$$

where the limits of integration are the y-coordinates of the points of intersection of the two curves. These are found by setting the x's equal to each other and solving for y.

$$\begin{aligned} x &= x \\ y^2 &= y + 2 \\ y^2 - y - 2 &= 0 \\ (y+1)(y-2) &= 0 \\ y = -1,\ y &= 2 \end{aligned}$$

From the graph we see that the right curve is $x = y + 2$ and the left curve is $x = y^2$.

Therefore, the area is:

$$\begin{aligned}
\text{Area} &= \int_c^d (\text{right} - \text{left})\ dx \\
&= \int_{-1}^{2} \left[(y+2) - y^2\right]\ dy \\
&= \int_{-1}^{2} \left(-y^2 + y + 2\right)\ dy \\
&= \left[-\frac{1}{3}y^3 + \frac{1}{2}y^2 + 2y\right]_{-1}^{2} \\
&= \left[-\frac{1}{3}(2)^3 + \frac{1}{2}(2)^2 + 2(2)\right] - \left[-\frac{1}{3}(-1)^3 + \frac{1}{2}(-1)^2 + 2(-1)\right] \\
&= \boxed{\frac{9}{2}}
\end{aligned}$$