Math 181, Exam 1, Fall 2012
Problem 1 Solution

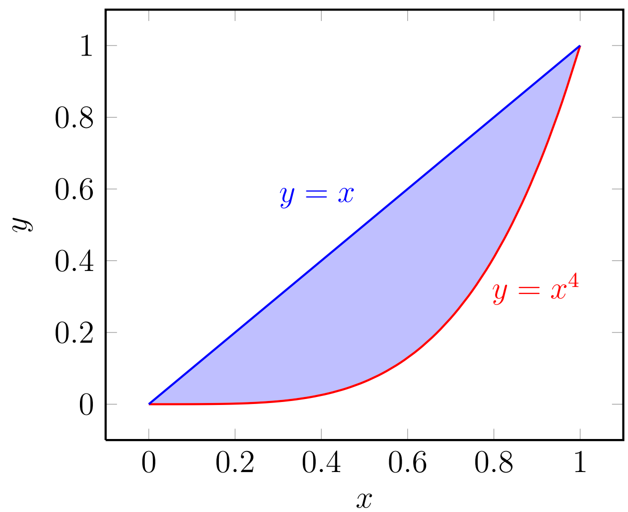
1. Let R be the region enclosed by the curves $y = x^4$ and $y = x$.

   (a) Sketch the region R.

   (b) Write down the integral representing the volume of the solid obtained by revolving R about the x-axis.

   (c) Compute the volume of the solid.

Solution:



(a)

(b) The solid of revolution has cross-sections that are washers. We then use the formula:

$$V = \int_a^b \pi \left[f(x)^2 - g(x)^2 \right] \, dx$$

where $f(x) = x$ and $g(x) = x^4$ and the interval is $[a, b] = [0, 1]$. Thus, the volume integral is:

$$V = \int_0^1 \pi \left(x^2 - x^8 \right) \, dx$$

(c) The volume calculation is as follows:

$$V = \pi \left[\frac{x^3}{3} - \frac{x^9}{9} \right]_0^1$$

$$V = \pi \left[\frac{1}{3} - \frac{1}{9} \right]$$

$$V = \frac{2\pi}{9}$$

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Problem 2 Solution

2. Compute the arc length of the curve given by $y(x) = 5 - 2x^{3/2}$ between $x = 0$ and $x = 9$.

Solution: The arc length is computed via the formula

$$L = \int_a^b \sqrt{1 + y'(x)^2}\, dx$$

Since $y(x) = 5 - 2x^{3/2}$ we know that $y'(x) = -3x^{1/2}$ and $y'(x)^2 = 9x$. Therefore, the arc length is

$$L = \int_0^9 \sqrt{1 + 9x}\, dx$$
$$L = \left[\frac{2}{27}(1 + 9x)^{3/2}\right]_0^9$$
$$L = \left[\frac{2}{27}(1 + 9 \cdot 9)^{3/2}\right] - \left[\frac{2}{27}(1 + 9 \cdot 0)^{3/2}\right]$$
$$L = \frac{2}{27}(82^{3/2} - 1)$$

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Problem 3 Solution

3. Compute the following integrals:

(a) $\displaystyle\int x^3 \sin(x^2)\,dx$

(b) $\displaystyle\int x^2 4^x\,dx$

Solution:

(a) We begin by letting $u = x^2$, $\frac{1}{2}\,du = x\,dx$. Using these substitutions, the integral is transformed as follows:

$$\begin{aligned}\int x^3 \sin(x^2)\,dx &= \int x^2 \sin(x^2)\cdot x\,dx \\ &= \int u\sin(u)\cdot\frac{1}{2}\,du \\ &= \frac{1}{2}\int u\sin(u)\,du\end{aligned}$$

The resulting integral may be evaluated using integration by parts. Letting $w = u$ and $dv = \sin(u)\,du$ we have $dw = du$ and $v = -\cos(u)$. Thus, using the integration by parts formula

$$\int w\,dv = wv - \int v\,dw$$

we obtain

$$\begin{aligned}\int u\sin(u)\,du &= u(-\cos(u)) - \int(-\cos(u))\,du \\ &= -u\cos(u) + \int \cos(u)\,du \\ &= -u\cos(u) + \sin(u) + C\end{aligned}$$

Finally, we use the fact that $u = x^2$ to write our answer as

$$\int x^3\sin(x^2)\,dx = -x^2\cos(x^2) + \sin(x^2) + C$$

(b) The integral may be evaluated using tabular integration.

D	I	sign
x^2	4^x	
$2x$	$\frac{1}{\ln(4)} \cdot 4^x$	$+$
2	$\frac{1}{(\ln(4))^2} \cdot 4^x$	$-$
0	$\frac{1}{(\ln(4))^3} \cdot 4^x$	$+$
0	$\frac{1}{(\ln(4))^4} \cdot 4^x$	$-$

$$\int x^2 4^x \, dx = +\frac{1}{\ln(4)} \cdot 4^x \cdot x^2 \; - \frac{1}{(\ln(4))^2} \cdot 4^x \cdot 2x \; + \frac{1}{(\ln(4))^3} \cdot 4^x \cdot 2 + C$$

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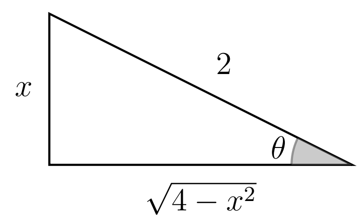
Problem 4 Solution

4. Compute the integral $\int \frac{dx}{(4-x^2)^{3/2}}$.

Solution: The computation requires the trigonometric substitution $x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$. Plugging these into the integral gives us

$$\begin{aligned}
\int \frac{dx}{(4-x^2)^{3/2}} &= \int \frac{2\cos\theta\,d\theta}{(4-(2\sin\theta)^2)^{3/2}} \\
&= \int \frac{2\cos\theta}{(4-4\sin^2\theta)^{3/2}}\,d\theta \\
&= \int \frac{2\cos\theta}{(4\cos^2\theta)^{3/2}}\,d\theta \\
&= \int \frac{2\cos\theta}{4^{3/2}(\cos^2\theta)^{3/2}}\,d\theta \\
&= \int \frac{2\cos\theta}{8\cos^3\theta}\,d\theta \\
&= \frac{1}{4}\int \sec^2\theta\,d\theta \\
&= \frac{1}{4}\tan\theta + C
\end{aligned}$$

Since $x = 2\sin\theta$ we know that $\sin\theta = \frac{x}{2}$. We can then construct a right triangle where the side opposite the angle θ is x and the hypotenuse is 2. Using the Pythagorean Theorem, the side adjacent to θ is $\sqrt{4-x^2}$.



Thus, the tangent of θ is the ratio of the side opposite θ to the adjacent side.

$$\tan\theta = \frac{x}{\sqrt{4-x^2}}$$

The integral is then

$$\int \frac{dx}{(4-x^2)^{3/2}} = \frac{1}{4} \cdot \frac{x}{\sqrt{4-x^2}} + C$$

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Problem 5 Solution

5. Compute the following integrals:

(a) $\displaystyle\int \frac{5}{x^2+3x-4}\,dx$,

(b) $\displaystyle\int \frac{dx}{x^2+4x+5}\,dx$.

Solution:

(a) The integrand is a rational function and the denominator factors into $(x+4)(x-1)$. Thus, we may use the method of partial fraction decomposition. Since the denominator has two distinct roots we decompose the integrand as follows:

$$\frac{5}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}.$$

Clearing denominators gives us

$$5 = A(x-1) + B(x+4).$$

Letting $x=1$ leads to $B=1$ and letting $x=-4$ leads to $A=-1$. Replacing A and B in the decomposition and evaluating the integral gives us:

$$\int \frac{5}{x^2+3x-4}\,dx = \int \left(\frac{-1}{x+4} + \frac{1}{x-1}\right)dx,$$

$$\boxed{\int \frac{5}{x^2+3x-4}\,dx = -\ln|x+4| + \ln|x-1| + C.}$$

(b) Once again, the integrand is a rational function but the denominator does not factor nicely. Therefore, we resort to completing the square:

$$x^2+4x+5 = (x+2)^2+1$$

We then introduce the substitution $u=x+2$, $du=dx$ to convert the integral into:

$$\int \frac{dx}{x^2+4x+5} = \int \frac{du}{u^2+1} = \arctan(u) + C.$$

Replacing u with $x+2$ gives us our result:

$$\boxed{\int \frac{dx}{x^2+4x+5} = \arctan(x+2) + C}$$