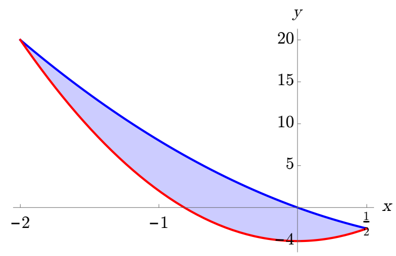
Math 181, Exam 1, Fall 2014
Problem 1 Solution

1. Consider the functions

$$f(x) = 2x^2 - 6x \quad \text{and} \quad g(x) = 6x^2 - 4.$$

   a. Find the x-coordinate of the intersection points of these two graphs.

   b. Compute the area of the region bounded by the graphs of the functions $f(x)$ and $g(x)$.



Solution:

a. The x-coordinates of the intersection points are solutions to the equation $f(x) = g(x)$.

$$\begin{aligned} 2x^2 - 6x &= 6x^2 - 4 \\ -4x^2 - 6x + 4 &= 0 \\ -2(2x^2 + 3x - 2) &= 0 \\ 2x^2 + 3x - 2 &= 0 \\ (x+2)(2x-1) &= 0 \\ x = -2, \quad x &= \frac{1}{2} \end{aligned}$$

b. Since $f(x) \geq g(x)$ on the interval $[-2, \frac{1}{2}]$, the area enclosed by the curves is:

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]\,dx \\ A &= \int_{-2}^{1/2} \left[(2x^2 - 6x) - (6x^2 - 4)\right]\,dx \\ A &= \int_{-2}^{1/2} \left(-4x^2 - 6x + 4\right)\,dx \\ A &= \left[-\frac{4}{3}x^3 - 3x^2 + 4x\right]_{-2}^{1/2} \\ A &= \left[-\frac{4}{3}\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)\right] - \left[-\frac{4}{3}(-2)^3 - 3(-2)^2 + 4(-2)\right] \\ A &= \frac{125}{12} \end{aligned}$$

Math 181, Exam 1, Fall 2014
Problem 2 Solution

2. Evaluate the following integrals. Be sure to specify which integration method(s) you are using and to show your work!

a. $\displaystyle\int x^2 e^{2x}\,dx$

b. $\displaystyle\int \frac{x+1}{x^2+9}\,dx$

Solution:

a. We use integration by parts to evaluate the integral. The corresponding formula is:

$$\int u\,dv = uv - \int v\,du$$

Letting $u = x^2$ and $dv = e^{2x}\,dx$ yields $u = 2x\,dx$ and $v = \frac{1}{2}e^{2x}$. Thus, we have

$$\int x^2 e^{2x}\,dx = \frac{1}{2}x^2 e^{2x} - \frac{1}{2}\int xe^{2x}\,dx$$

The integral on the right hand side is evaluated using integration by parts. Letting $u = x$ and $dv = e^{2x}\,dx$ yields $du = dx$ and $v = \frac{1}{2}e^{2x}$. Thus, we have

$$\int x^2 e^{2x} = \frac{1}{2}x^2 e^{2x} - \left(\frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx\right)$$
$$\int x^2 e^{2x} = \frac{1}{2}x^2 e^{2x} - \left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right) + C$$
$$\int x^2 e^{2x} = \frac{1}{2}x^2 e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$$

b. We begin by splitting the integrand into a sum of two rational functions and then use the sum rule for integrals.

$$\int \frac{x+1}{x^2+9}\,dx = \int \frac{x}{x^2+9}\,dx + \int \frac{1}{x^2+9}\,dx \qquad (1)$$

The first integral on the right hand side of Equation (1) may be evaluated using the substitution $u = x^2 + 9$, $\frac{1}{2}\,du = x\,dx$. These substitutions yield the result:

$$\int \frac{x}{x^2+9}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2+9|$$

An algebraic manipulation of the second integral yields:

$$\int \frac{1}{x^2+9}\,dx = \int \frac{1}{9(\frac{x^2}{9}+1)}\,dx = \frac{1}{9}\int \frac{1}{(\frac{x}{3})^2+1}\,dx$$

Using the substitution $u = \frac{x}{3}$, $3\,du = dx$ yields the result:

$$\int \frac{1}{x^2+9}\,dx = \frac{1}{9}\int \frac{1}{u^2+1}\,(3\,du) = \frac{1}{3}\int \frac{1}{u^2+1}\,du = \frac{1}{3}\arctan(u) = \frac{1}{3}\arctan\left(\frac{x}{3}\right)$$

Thus, the integral is:

$$\int \frac{x+1}{x^2+9}\,dx = \frac{1}{2}\ln|x^2+9| + \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

Math 181, Exam 1, Fall 2014
Problem 3 Solution

3.

a. Write the general formula for the length of the graph of a function $f(x)$ between $x = a$ and $x = b$.

b. Find the length of the graph of the function $f(x) = \frac{2}{x} + \frac{x^3}{24}$ between $x = 1$ and $x = 3$.

Solution:

a. The arclength of $f(x)$ on $[a, b]$ is:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

b. The derivative $f'(x)$ is:

$$f'(x) = -\frac{2}{x^2} + \frac{x^2}{8}$$

The expression $\sqrt{1 + f'(x)^2}$ simplifies as follows:

$$\begin{aligned}
\sqrt{1 + f'(x)^2} &= \sqrt{1 + \left(-\frac{2}{x^2} + \frac{x^2}{8}\right)^2} \\
&= \sqrt{1 + \frac{4}{x^4} - \frac{1}{2} + \frac{x^4}{64}} \\
&= \sqrt{\frac{4}{x^4} + \frac{1}{2} + \frac{x^4}{64}} \\
&= \sqrt{\left(\frac{2}{x^2} + \frac{x^2}{8}\right)^2} \\
&= \frac{2}{x^2} + \frac{x^2}{8}
\end{aligned}$$

The arclength of $f(x)$ on $[1, 3]$ is:

$$\begin{aligned}
L &= \int_1^3 \sqrt{1 + f'(x)^2}\, dx \\
L &= \int_1^3 \left(\frac{2}{x^2} + \frac{x^2}{8}\right) dx \\
L &= \left[-\frac{2}{x} + \frac{x^3}{24}\right]_1^3 \\
L &= \left[-\frac{2}{3} + \frac{3^3}{24}\right] - \left[-\frac{2}{1} + \frac{1^3}{24}\right] \\
L &= \frac{29}{12}
\end{aligned}$$

Math 181, Exam 1, Fall 2014
Problem 4 Solution

4. Let R be the region bounded by the graphs of the functions

$$y = x^4 \quad \text{and} \quad y = \sqrt{x}$$

and consider the solid of revolution obtained by revolving R about the x-axis.

a. Compute the volume V of this solid of revolution using slices.

b. Compute the volume V of this solid of revolution using shells.

Solution: The curves intersect when $x^4 = \sqrt{x}$. After squaring both sides we obtain $x^8 = x$. Subtracting x from both sides of the equation and factoring out an x yields $x(x^7 - 1) = 0$. Thus, the two real solutions are $x = 0$ and $x = 1$. The coordinates of the intersection points are $(0, 0)$ and $(1, 1)$. Moreover, we have $\sqrt{x} \geq x^4$ on the interval $0 \leq x \leq 1$.

a. The cross-sections of this solid are washers. The corresponding volume formula is:

$$V = \int_a^b \pi \left[f(x)^2 - g(x)^2 \right] \, dx$$

where $a = 0$, $b = 1$, $f(x) = \sqrt{x}$, and $g(x) = x^4$. Thus, the volume V is:

$$V = \int_0^1 \pi \left[\left(\sqrt{x}\right)^2 - \left(x^4\right)^2 \right] \, dx$$

$$V = \pi \int_0^1 \left(x - x^8\right) \, dx$$

$$V = \pi \left[\frac{1}{2}x^2 - \frac{1}{9}x^9 \right]_0^1$$

$$V = \pi \left(\frac{1}{2} - \frac{1}{9} \right)$$

$$V = \frac{7\pi}{18}$$

b. In order to use shells we write the equations for the bounding curves as $x = y^2$ and $x = y^{1/4}$. The volume formula for the shell method is:

$$V = \int_c^d 2\pi y \left[p(y) - g(y) \right] \, dy$$

where $c = 0$, $d = 1$, $p(y) = y^{1/4}$, and $q(y) = y^2$. Thus, the volume is:

$$
\begin{aligned}
V &= \int_0^1 2\pi y \left(y^{1/4} - y^2\right)\, dy \\
V &= \int_0^1 2\pi \left(y^{5/4} - y^3\right)\, dy \\
V &= 2\pi \left[\frac{4}{9}y^{9/4} - \frac{1}{4}y^4\right]_0^1 \\
V &= 2\pi \left(\frac{4}{9} - \frac{1}{4}\right) \\
V &= \frac{7\pi}{18}
\end{aligned}
$$

Math 181, Exam 1, Fall 2014
Problem 5 Solution

5. Evaluate the following integrals. Be sure to specify which integration method(s) you are using and to show your work!

a. $\displaystyle\int x\ln(x)\,dx$

b. $\displaystyle\int x^3\sin(x^2)\,dx$

Solution:

a. We use integration by parts to evaluate the integral. The corresponding formula is:

$$\int u\,dv = uv - \int v\,du$$

Letting $u = \ln(x)$ and $dv = x\,dx$ yields $u = \frac{1}{x}\,dx$ and $v = \frac{1}{2}x^2$. Thus, we have

$$\begin{aligned}\int x\ln(x)\,dx &= \frac{1}{2}x^2\ln(x) - \int \frac{1}{2}x^2\cdot\frac{1}{x}\,dx\\ &= \frac{1}{2}x^2\ln(x) - \frac{1}{2}\int x\,dx\\ &= \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C\end{aligned}$$

b. We begin by rewriting the integral as follows:

$$\int x^3\sin(x^2)\,dx = \int x\cdot x^2\sin(x^2)\,dx$$

We use the substitution $u = x^2$, $\frac{1}{2}\,du = x\,dx$ to obtain the result:

$$\int x^3\sin(x^2)\,dx = \frac{1}{2}\int u\sin(u)\,du$$

The integral on the right hand side above must be computed using integration by parts. Letting $w = u$ and $dv = \sin(u)\,du$ yields $dw = du$ and $v = -\cos(u)$. The integration by parts formula:

$$\int w\,dv = wv - \int v\,dw$$

yields the result:

$$
\begin{aligned}
\int x^3 \sin(x^2)\, dx &= \frac{1}{2} \int u \sin(u)\, du \\
&= \frac{1}{2} \left[-u \cos(u) - \int (-\cos(u))\, du \right] \\
&= \frac{1}{2} \left(-u \cos(u) + \sin(u) \right) + C \\
&= \frac{1}{2} \big(-x^2 \cos(x^2) + \sin(x^2) \big) + C
\end{aligned}
$$

Math 181, Exam 1, Fall 2014
Problem 6 Solution

6. Consider the region bounded by the curves $y = (x-2)^2 - 1$, $x = 0$, and $y = 0$. Find the volume of revolution of the resulting solid, when the region is rotated about:

a. (4 points) the y-axis

b. (4 points) the axis $y = 4$.

You may use either slices or shells to compute these volumes, as you prefer.

Solution:

a. The volume of the solid obtained when rotating about the y-axis is best found using shells. The corresponding formula is

$$V = \int_a^b 2\pi x \, [f(x) - g(x)] \, dx.$$

The region is bounded on the left by $x = 0$, from below by $y = 0 = g(x)$, and from above by $y = (x-2)^2 - 1 = f(x)$. The latter curves intersect at $x = 1$. Thus, the limits of integration are $a = 0$ and $b = 1$. The volume is then

$$\begin{aligned} V &= \int_a^b 2\pi x \, [f(x) - g(x)] \, dx, \\ &= \int_0^1 2\pi x \left[(x-2)^2 - 1 - 0\right] dx, \\ &= 2\pi \int_0^1 x \left(x^2 - 4x + 3\right) dx, \\ &= 2\pi \int_0^1 \left(x^3 - 4x^2 + 3x\right) dx, \\ &= 2\pi \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_0^1, \\ &= 2\pi \left[\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right], \\ &= \frac{5\pi}{6}. \end{aligned}$$

b. The volume of the solid obtained when rotating about the axis $y = 4$ is best found using washers. The inner radii of the washers are given by $r = 4 - \left[(x-2)^2 - 1\right]$ and the outer radii of the washers are given by $R = 4$. Using the same limits of integration as in part a., the volume of this solid is

$$
\begin{aligned}
V &= \int_a^b \pi(R^2 - r^2)\,dx, \\
&= \int_0^1 \pi \left[4^2 - \left(4 - \left((x-2)^2 - 1\right)\right)^2\right]\,dx, \\
&= \pi \int_0^1 \left[16 - \left(4 - (x^2 - 4x + 3)\right)^2\right]\,dx, \\
&= \pi \int_0^1 \left[16 - \left(-x^2 + 4x + 1)\right)^2\right]\,dx, \\
&= \pi \int_0^1 \left[16 - \left(x^4 - 8x^3 + 14x^2 + 8x + 1\right)\right]\,dx, \\
&= \pi \int_0^1 \left(-x^4 + 8x^3 - 14x^2 - 8x + 15\right)\,dx, \\
&= \pi \left[-\frac{1}{5}x^5 + 2x^4 - \frac{14}{3}x^3 - 4x^2 + 15x\right]_0^1, \\
&= \pi \left[-\frac{1}{5} + 2 - \frac{14}{3} - 4 + 15\right], \\
&= \frac{122\pi}{15}.
\end{aligned}
$$