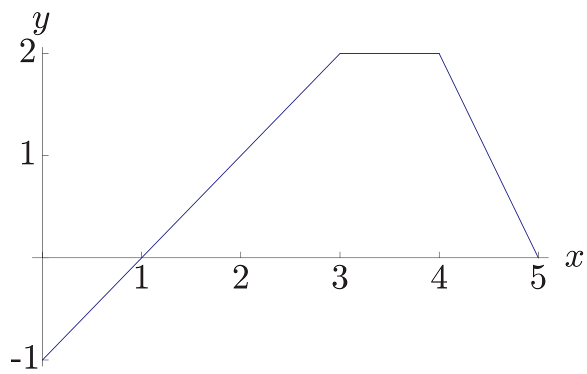
Math 181, Exam 1, Spring 2006
Problem 1 Solution

1. The graph of a function $g(x)$ is given below. Let G be an antiderivative for g on the interval $[0,5]$ with $G(1) = 2$. Compute $G(0)$, $G(3)$, and $G(5)$.



Solution: Since G is an antiderivative of g we know that $G'(x) = g(x)$. That is,

$$G(x) = \int g(x)\,dx$$

where

$$g(x) = \begin{cases} x-1 & \text{if } 0 \le x < 3 \\ 2 & \text{if } 3 \le x < 4 \\ -2x+10 & \text{if } 4 \le x \le 5 \end{cases}$$

- On the interval $0 \le x < 3$, we have $g(x) = x - 1$. Therefore,

$$G(x) = \int (x-1)\,dx$$
$$G(x) = \frac{1}{2}x^2 - x + C_1$$

The value of C_1 is found by using the fact that $G(1) = 2$.

$$G(1) = 2$$
$$\frac{1}{2}(1)^2 - 1 + C_1 = 2$$
$$C_1 = \frac{5}{2}$$

Therefore, on the interval $0 \le x < 3$ we have $G(x) = \frac{1}{2}x^2 - x + \frac{5}{2}$ and, thus, $\boxed{G(0) = \frac{5}{2}}$.

- On the interval $3 \leq x < 4$, we have $g(x) = 2$. Therefore,

$$G(x) = \int 2\,dx$$
$$G(x) = 2x + C_2$$

The value of C_2 is found by ensuring continuity of $G(x)$ at $x = 3$. That is, we need:

$$\lim_{x\to 3^-} G(x) = \lim_{x\to 3^+} G(x)$$
$$\lim_{x\to 3^-} \left(\frac{1}{2}x^2 - x + \frac{5}{2}\right) = \lim_{x\to 3^+} (2x + C_2)$$
$$\frac{1}{2}(3)^2 - 3 + \frac{5}{2} = 2(3) + C_2$$
$$C_2 = -2$$

Therefore, on the interval $3 \leq x < 4$ we have $G(x) = 2x - 2$ and, thus, $\boxed{G(3) = 4}$.

- On the interval $4 \leq x \leq 5$, we have $g(x) = -2x + 10$. Therefore,

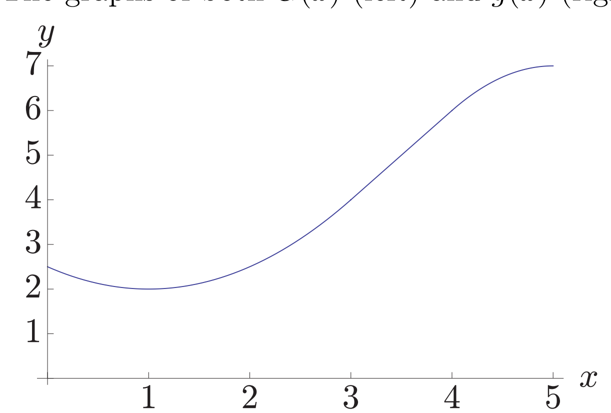
$$G(x) = \int (-2x + 10)\,dx$$
$$G(x) = -x^2 + 10x + C_3$$

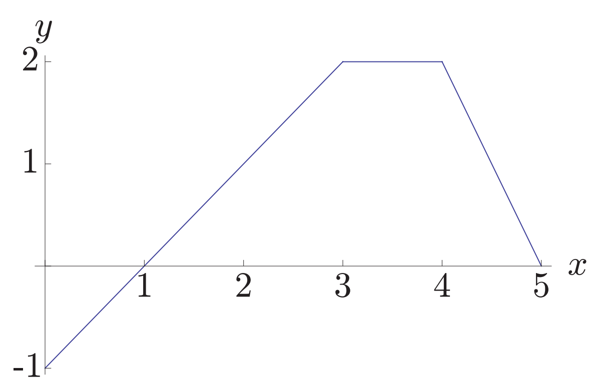
The value of C_3 is found by ensuring continuity of $G(x)$ at $x = 4$. That is, we need:

$$\lim_{x\to 4^-} G(x) = \lim_{x\to 4^+} G(x)$$
$$\lim_{x\to 4^-} (2x - 2) = \lim_{x\to 4^+} (-x^2 + 10x + C_3)$$
$$2(4) - 2 = -(4)^2 + 10(4) + C_3$$
$$C_3 = -18$$

Therefore, on the interval $4 \leq x \leq 5$ we have $G(x) = -x^2 + 10x - 18$ and, thus, $\boxed{G(5) = 7}$.

The graphs of both $G(x)$ (left) and $g(x)$ (right) are shown below.





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Problem 2 Solution

2. On the planet Penthesilea IV, the gravitational acceleration is -20 ft/sec^2. A stone is thrown upwards from a height of 60 ft with initial velocity 100 ft/sec.

i) When will the stone reach its maximum height?

ii) What is the maximum height reached by the stone?

Solution:

i) The velocity of the stone is given by:

$$v(t) = -gt + v_0,$$

where $g = -20$ ft/sec^2 is the gravitational acceleration and $v_0 = 100$ ft/sec is the initial velocity. When the stone reaches its maximum height, its velocity is 0. The time when this happens is then:

$$\begin{aligned} v(t) &= 0 \\ -20t + 100 &= 0 \\ & \boxed{t = 5 \text{ sec}} \end{aligned}$$

ii) The height of the stone is given by:

$$x(t) = -\frac{1}{2}gt^2 + v_0 t + x_0$$

where $x_0 = 60$ ft is the initial height. Evaluating $x(t)$ at $t = 5$ sec we get:

$$x(5) = -\frac{1}{2}(20)(5)^2 + 100(5) + 60 = \boxed{310 \text{ ft}}$$

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Problem 3 Solution

3. Differentiate the function:

$$T(x) = \int_1^{\cos x} e^{t^2}\,dt$$

Solution: Using the Fundamental Theorem of Calculus Part II and the Chain Rule, the derivative $T'(x)$ is:

$$\begin{aligned} T'(x) &= \frac{d}{dx}\int_1^{\cos x} e^{t^2}\,dt \\ &= e^{(\cos x)^2} \cdot \frac{d}{dx}\cos x \\ &= \boxed{e^{(\cos x)^2} \cdot (-\sin x)} \end{aligned}$$

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Problem 4 Solution

4. Compute the definite integral:

$$\int_0^1 xe^{2x}\,dx$$

Solution: We will evaluate the integral using Integration by Parts. Let $u = x$ and $v' = e^{2x}$. Then $u' = 1$ and $v = \frac{1}{2}e^{2x}$. Using the Integration by Parts formula:

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

we get:

$$\begin{aligned}
\int_0^1 xe^{2x}\,dx &= \left[\frac{1}{2}xe^{2x}\right]_0^1 - \int_0^1 \frac{1}{2}e^{2x}\,dx \\
&= \left[\frac{1}{2}xe^{2x}\right]_0^1 - \left[\frac{1}{4}e^{2x}\right]_0^1 \\
&= \left(\frac{1}{2}e^2 - 0\right) - \left(\frac{1}{4}e^2 - \frac{1}{4}\right) \\
&= \boxed{\frac{1}{4}e^2 + \frac{1}{4}}
\end{aligned}$$

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Problem 5 Solution

5. Find:

$$\int \frac{x}{\sqrt{x+4}}\,dx$$

Solution: We will evaluate the integral using the u-substitution method. Let $u = x+4$ so that $du = dx$ and $x = u-4$. Substituting these expressions into the given integral and evaluating we get:

$$\begin{aligned}
\int \frac{x}{\sqrt{x+4}}\,dx &= \int \frac{u-4}{\sqrt{u}}\,du \\
&= \int \left(\frac{u}{\sqrt{u}} - \frac{4}{\sqrt{u}}\right) du \\
&= \int \left(u^{1/2} - 4u^{-1/2}\right) du \\
&= \frac{2}{3}u^{3/2} - 8u^{1/2} + C \\
&= \boxed{\frac{2}{3}(x+4)^{3/2} - 8(x+4)^{1/2} + C}
\end{aligned}$$

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Problem 6 Solution

6. Find:

$$\int \frac{dx}{x^2 - 3x + 2}$$

Solution: We will evaluate the integral using Partial Fraction Decomposition. First, we factor the denominator and then decompose the rational function into a sum of simpler rational functions.

$$\frac{1}{x^2 - 3x + 2} = \frac{1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

Next, we multiply the above equation by $(x-1)(x-2)$ to get:

$$1 = A(x-2) + B(x-1)$$

Then we plug in two different values for x to create a system of two equations in two unknowns (A,B). We select $x = 1$ and $x = 2$ for simplicity.

$$\begin{aligned} x = 1: &\quad A(1-2) + B(1-1) = 1 \quad \Rightarrow \quad A = -1 \\ x = 2: &\quad A(2-2) + B(2-1) = 1 \quad \Rightarrow \quad B = 1 \end{aligned}$$

Finally, we plug these values for A and B back into the decomposition and integrate.

$$\begin{aligned} \int \frac{dx}{x^2 - 3x + 2} &= \int \left(\frac{A}{x-1} + \frac{B}{x-2} \right) dx \\ &= \int \left(\frac{-1}{x-1} + \frac{1}{x-2} \right) dx \\ &= \boxed{-\ln|x-1| + \ln|x-2| + C} \end{aligned}$$

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Problem 7 Solution

7. Compute the definite integral:

$$\int x^6 \ln x \, dx$$

Solution: We will evaluate the integral using Integration by Parts. Let $u = \ln x$ and $v' = x^6$. Then $u' = \frac{1}{x}$ and $v = \frac{1}{7}x^7$. Using the Integration by Parts formula:

$$\int uv' \, dx = uv - \int u'v \, dx$$

we get:

$$\begin{aligned}
\int x^6 \ln x \, dx &= \frac{1}{7}x^7 \ln x - \int \left(\frac{1}{x}\right)\left(\frac{1}{7}x^7\right) dx \\
&= \frac{1}{7}x^7 \ln x - \frac{1}{7}\int x^6 \, dx \\
&= \boxed{\frac{1}{7}x^7 \ln x - \frac{1}{49}x^7 + C}
\end{aligned}$$

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Problem 8 Solution

8. Find:

$$\int \arctan x \, dx$$

Solution: We will evaluate the integral using Integration by Parts. Let $u = \arctan x$ and $v' = 1$. Then $u' = \dfrac{1}{x^2+1}$ and $v = x$. Using the Integration by Parts formula:

$$\int uv' \, dx = uv - \int u'v \, dx$$

we get:

$$\int \arctan x \, dx = x \arctan x - \int \frac{1}{x^2+1} \, x \, dx.$$

Use the substitution $w = x^2 + 1$ to evaluate the integral on the right hand side. Then $dw = 2x \, dx \quad \Rightarrow \quad \frac{1}{2} dw = x \, dx$ and we get:

$$\begin{aligned} \int \arctan x \, dx &= x \arctan x - \int \frac{x}{x^2+1} \, dx \\ &= x \arctan x - \frac{1}{2} \int \frac{1}{w} \, dw \\ &= x \arctan x - \frac{1}{2} \ln |w| + C \\ &= \boxed{x \arctan x - \frac{1}{2} \ln(x^2+1) + C} \end{aligned}$$

Note that the absolute value signs aren't needed because $x^2 + 1 > 0$ for all x.