Math 181, Exam 1, Spring 2008
Problem 1 Solution

1. Consider the region enclosed by the curves $y = 2x$ and $x + y^2 = 3$.

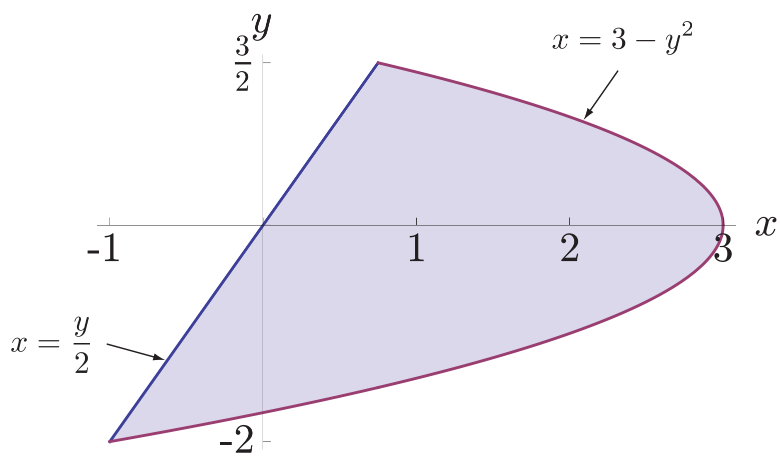
   (a) Sketch the region.

   (b) Find the points of intersection (determine both the x and y coordinates for each point).

   (c) Compute the area enclosed by the curves.

Solution:

(a) The region is sketched below.



(b) The points of intersection are found by plugging $y = 2x$ into $x + y^2 = 3$ and solving for x.

$$\begin{aligned} x + y^2 &= 3 \\ x + (2x)^2 &= 3 \\ 4x^2 + x - 3 &= 0 \\ (4x-3)(x+1) &= 0 \\ x &= \frac{3}{4},\ x = -1 \end{aligned}$$

The corresponding y-values are found by plugging the above x-values into the equation $y = 2x$. Therefore,

$$\begin{aligned} x = \frac{3}{4} :\ & y = 2x = 2\left(\frac{3}{4}\right) = \frac{3}{2} \\ x = -1 :\ & y = 2x = 2(-1) = -2 \end{aligned}$$

(c) The formula we use to compute the area of the region is:

$$\text{Area} = \int_c^d (\text{right} - \text{left})\ dy$$

where c and d are the y-coordinates of the points of intersection of the two curves. From the graph we see that the right curve is $x = 3 - y^2$ and the left curve is $x = \dfrac{y}{2}$. The limits of integration are $c = -2$ and $d = \dfrac{3}{2}$, as found in part (b). Therefore, the area is:

$$\begin{aligned}
\text{Area} &= \int_c^d (\text{right} - \text{left})\ dy \\
&= \int_{-2}^{3/2} \left[(3 - y^2) - \frac{y}{2}\right]\ dy \\
&= \left[3y - \frac{1}{3}y^3 - \frac{1}{4}y^2\right]_{-2}^{3/2} \\
&= \left[3\left(\frac{3}{2}\right) - \frac{1}{3}\left(\frac{3}{2}\right)^3 - \frac{1}{4}\left(\frac{3}{2}\right)^2\right] - \left[3(-2) - \frac{1}{3}(-2)^3 - \frac{1}{4}(-2)^2\right] \\
&= \left[\frac{9}{2} - \frac{9}{8} - \frac{9}{16}\right] - \left[-6 + \frac{8}{3} - 1\right] \\
&= \boxed{\frac{343}{48}}
\end{aligned}$$

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Problem 2 Solution

2. Compute the following indefinite integrals.

(a) $\displaystyle\int x\sqrt{x-1}\,dx$

(b) $\displaystyle\int \frac{5}{\sqrt{1-25x^2}}\,dx$

Solution:

(a) The integral is computed using the u-substitution method. Let $u = x - 1$. Then $du = dx$ and $x = u + 1$. Substituting these into the integral and evaluating we get:

$$\begin{aligned}
\int x\sqrt{x-1}\,dx &= \int (u+1)\sqrt{u}\,du \\
&= \int \left(u^{3/2} + u^{1/2}\right)\,du \\
&= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C \\
&= \boxed{\frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C}
\end{aligned}$$

(b) The integral is computed using the u-substitution method. Let $u = 5x$. Then $du = 5\,dx$ and we get:

$$\begin{aligned}
\int \frac{5}{\sqrt{1-25x^2}}\,dx &= \int \frac{5}{\sqrt{1-(5x)^2}}\,dx \\
&= \int \frac{1}{\sqrt{1-u^2}}\,du \\
&= \arcsin u + C \\
&= \boxed{\arcsin(5x) + C}
\end{aligned}$$

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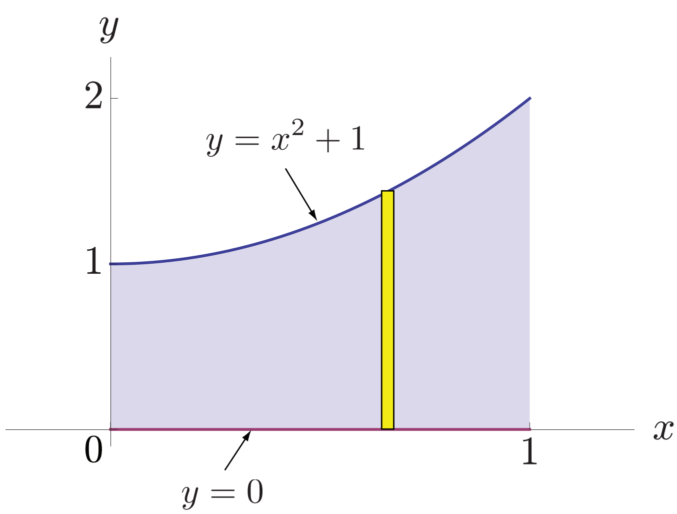
Problem 3 Solution

3. Find the volume of the solid of revolution obtained by rotating the region under the curve $y = x^2 + 1$ over the interval $0 \le x \le 1$:

(a) about the x-axis,

(b) about the y-axis

Solution:



(a) We find the volume of the solid obtained by rotating about the x-axis using the **Disk Method**. We use the Disk Method because the region is bounded below by the x-axis. In this case, the variable of integration is x and the corresponding formula is:

$$V = \pi \int_a^b f(x)^2 \, dx$$

where $f(x) = x^2 + 1$. The volume is then:

$$\begin{aligned} V &= \pi \int_0^1 \left(x^2 + 1\right)^2 \, dx \\ &= \pi \int_0^1 \left(x^4 + 2x^2 + 1\right) \, dx \\ &= \pi \left[\frac{1}{5}x^5 + \frac{2}{3}x^3 + x\right]_0^1 \\ &= \pi \left[\frac{1}{5} + \frac{2}{3} + 1\right] \\ &= \boxed{\frac{28\pi}{15}} \end{aligned}$$

(b) We find the volume of the solid obtained by rotating about the y-axis using the **Shell Method**. In this case, the variable of integration is x and the corresponding formula is:

$$V = 2\pi \int_a^b x\,(\text{top} - \text{bottom})\;dx$$

The top curve is $y = x^2 + 1$ and the bottom curve is $y = 0$. The volume is then:

$$\begin{aligned}
V &= 2\pi \int_0^1 x(x^2 + 1 - 0)\,dx \\
&= 2\pi \int_0^1 \left(x^3 + x\right)\,dx \\
&= 2\pi \left[\frac{1}{4}x^4 + \frac{1}{2}x^2\right]_0^1 \\
&= 2\pi \left[\frac{1}{4} + \frac{1}{2}\right] \\
&= \boxed{\frac{3\pi}{2}}
\end{aligned}$$

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Problem 4 Solution

4. Compute each of the following:

(a) $\displaystyle\int_1^4 \left(\frac{1}{x} - x^{3/2}\right)\, dx$

(b) $\displaystyle\frac{d}{dx}\int_1^{x^2} \tan\left(\sqrt{t}\right)\, dt$

Solution:

(a) Using the Fundamental Theorem of Calculus Part I, the value of the integral is:

$$\begin{aligned}\int_1^4 \left(\frac{1}{x} - x^{3/2}\right)\, dx &= \left[\ln|x| - \frac{2}{5}x^{5/2}\right]_1^4 \\ &= \left[\ln|4| - \frac{2}{5}(4)^{5/2}\right] - \left[\ln|1| - \frac{2}{5}(1)^{5/2}\right] \\ &= \ln 4 - \frac{64}{5} - 0 + \frac{2}{5} \\ &= \boxed{\ln 4 - \frac{62}{5}}\end{aligned}$$

(b) Using the Fundamental Theorem of Calculus Part II and the Chain Rule, the derivative is:

$$\begin{aligned}F'(x) &= \frac{d}{dx}\int_1^{x^2} \tan\left(\sqrt{t}\right)\, dt \\ &= \tan\left(\sqrt{x^2}\right)\cdot\frac{d}{dx}(x^2) \\ &= \boxed{\tan(|x|)\cdot(2x)}\end{aligned}$$

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Problem 5 Solution

5. Suppose you have money invested in a bank account with an interest rate of k. Let y_0 denote the initial amount invested and $y(t)$ denote the amount in the account after t years. The interest is compounded continuously so that:

$$y(t) = y_0 e^{kt}$$

(a) At what interest rate should \$1000 be invested so that there is \$1500 in the account after 10 years?

(b) If the interest rate is 5%, how many years will it take for your initial investment to double?

Use the approximations $\ln 2 \approx 0.7$ and $\ln 3 \approx 1.1$ to write your answer to (a) as an integer percent (for example, 8%) and your answer to (b) as an integer.

Solution:

(a) The initial amount invested is $y_0 = 1000$. After 10 years we have $y(10) = 1500$. Using the above formula and solving for k we have:

$$\begin{aligned} y(10) &= y_0 e^{k(10)} \\ 1500 &= 1000 e^{10k} \\ \frac{1500}{1000} &= e^{10k} \\ e^{10k} &= \frac{3}{2} \\ 10k &= \ln \frac{3}{2} \\ k &= \frac{1}{10} \ln \frac{3}{2} \end{aligned}$$

In order to use the approximations to turn our answer into a percentage, we must use the logarithm rule:

$$\ln \frac{b}{a} = \ln b - \ln a$$

Therefore,

$$\begin{aligned} k &= \frac{1}{10} \ln \frac{3}{2} \\ &= \frac{1}{10} (\ln 3 - \ln 2) \\ &\approx \frac{1}{10} (1.1 - 0.7) \\ &\approx 0.04 \\ &\approx \boxed{4\%} \end{aligned}$$

(b) Using $k = 0.05$, we must find the value of t so that $y(t) = 2y_0$.

$$\begin{aligned} y(t) &= y_0 e^{0.05t} \\ 2y_0 &= y_0 e^{0.05t} \\ 2 &= e^{0.05t} \\ 0.05t &= \ln 2 \\ t &= 20 \ln 2 \\ &\approx 20(0.7) \\ &\approx \boxed{\text{14 years}} \end{aligned}$$