Math 181, Exam 1, Spring 2012
Problem 1 Solution

1. An object moves along a straight line with velocity function $v(t) = t^2 + 2t + 5$.

   (a) Calculate the displacement from $t = 0$ to $t = 3$.

   (b) Is the displacement calculated in part (a) equal to the total distance traveled by the object? Explain why or why not.

Solution:

(a) By definition, the displacement of an object moving with velocity $v(t)$ from time t_0 to t_1 is given by

$$\text{displacement} = \int_{t_0}^{t_1} v(t)\,dt\ .$$

Using the given velocity function and the given time interval we get

$$\begin{aligned}\text{displacement} &= \int_0^3 \left(t^2 + 2t + 5\right)\,dt,\\ &= \left[\frac{t^3}{3} + t^2 + 5t\right]_0^3,\\ &= \frac{3^3}{3} + 3^2 + 5(3),\\ &= 33\ .\end{aligned}$$

(b) By definition, the distance traveled by an object moving with velocity $v(t)$ from time t_0 to t_1 is given by

$$\text{distance traveled} = \int_{t_0}^{t_1} |v(t)|\,dt\ .$$

We should notice that the given velocity function can be written as

$$v(t) = t^2 + 2t + 5 = (t+1)^2 + 4$$

by completing the square and that this function is positive for all t. Therefore, $|v(t)| = v(t)$ and the distance traveled is the same as displacement.

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Problem 2 Solution

2. Set up (but **do not evaluate**) the integrals that compute the following quantities:

(a) The length of the curve $y = 3x^2 + 1$ between $x = 0$ and $x = 2$.

(b) The volume of the solid obtained by revolving around the y-axis the region bounded by $y = \sin(x)$, the x-axis, $x = \dfrac{\pi}{4}$, and $x = \dfrac{3\pi}{4}$.

Solution:

(a) The formula for calculating the length of a curve $y = f(x)$ on the interval $[a, b]$ is given by

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx \ .$$

Since $y = f(x) = 3x^2 + 1$ we have $f'(x) = 6x$. Therefore, the length of the curve can be computed using the formula

$$L = \int_0^2 \sqrt{1 + (6x)^2}\, dx = \int_0^2 \sqrt{1 + 36x^2}\, dx \ .$$

(b) We compute the volume of the solid of revolution using the Shell Method. The formula we will use is

$$V = 2\pi \int_a^b x f(x)\, dx$$

where $a = \frac{\pi}{4}$, $b = \frac{3\pi}{4}$, and $f(x) = \sin(x)$. Plugging these quantities into the formula we get

$$V = 2\pi \int_{\pi/4}^{3\pi/4} x \sin(x)\, dx \ .$$

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Problem 3 Solution

3. Let R be the region below the curve $y = \sqrt{4-x}$ and between $x = 1$ and $x = 4$.

(a) Calculate the area of R.

(b) Calculate the volume of the solid obtained by revolving R around the x-axis.

Solution:

(a) The area of R is

$$\begin{aligned} A &= \int_1^4 \sqrt{4-x}\,dx, \\ &= \left[-\frac{2}{3}(4-x)^{3/2}\right]_1^4, \\ &= \left[-\frac{2}{3}(4-4)^{3/2}\right] - \left[-\frac{2}{3}(4-1)^{3/2}\right], \\ &= 2\sqrt{3}. \end{aligned}$$

(b) We calculate the volume using the Disk Method. The formula we will use is

$$V = \pi \int_a^b f(x)^2\,dx$$

where $a = 1$, $b = 4$, and $f(x) = \sqrt{4-x}$. Plugging these quantities into the formula we get

$$\begin{aligned} V &= \pi \int_1^4 \left(\sqrt{4-x}\right)^2\,dx, \\ &= \pi \int_1^4 (4-x)\,dx, \\ &= \pi \left[4x - \frac{1}{2}x^2\right]_1^4, \\ &= \pi \left[\left(4(4) - \frac{1}{2}(4)^2\right) - \left(4(1) - \frac{1}{2}(1)^2\right)\right], \\ &= \frac{9\pi}{2}. \end{aligned}$$

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Problem 4 Solution

4. Calculate the indefinite integrals:

(a) $\displaystyle\int (4x+1)\ln(x)\,dx$

(b) $\displaystyle\int \tan^3(x)\sec^4(x)\,dx$

(c) $\displaystyle\int \frac{1}{(2+3x^2)^{3/2}}\,dx$

Solution:

(a) We use Integration by Parts to calculate the integral. Letting $u = \ln(x)$ and $dv = (4x+1)\,dx$ we get $du = \frac{1}{x}\,dx$ and $v = 2x^2 + x$. Using the Integration by Parts formula we get

$$\int u\,dv = uv - \int v\,du,$$

$$\int (4x+1)\ln(x)\,dx = \left(2x^2+x\right)\ln(x) - \int \left(2x^2+x\right)\frac{1}{x}\,dx,$$

$$\int (4x+1)\ln(x)\,dx = \left(2x^2+x\right)\ln(x) - \int (2x+1)\,dx$$

$$\int (4x+1)\ln(x)\,dx = \left(2x^2+x\right)\ln(x) - \left(x^2+x\right) + C$$

(b) We begin by rewriting $\sec^4(x)$ as

$$\sec^4(x) = \sec^2(x)\sec^2(x) = \left(1+\tan^2(x)\right)\sec^2(x).$$

The integral then becomes

$$\int \tan^3(x)\sec^4(x)\,dx = \int \tan^3(x)\left(1+\tan^2(x)\right)\sec^2(x)\,dx.$$

Now let $u = \tan(x)$ so that $du = \sec^2(x)\,dx$. Making the proper substitutions into the above integral we find that

$$\int \tan^3(x)\sec^4(x)\,dx = \int \tan^3(x)\left(1+\tan^2(x)\right)\sec^2(x)\,dx,$$

$$\int \tan^3(x)\sec^4(x)\,dx = \int u^3\left(1+u^2\right)\,du,$$

$$\int \tan^3(x)\sec^4(x)\,dx = \int \left(u^3+u^5\right)\,du,$$

$$\int \tan^3(x)\sec^4(x)\,dx = \frac{1}{4}u^4 + \frac{1}{6}u^6 + C,$$

$$\int \tan^3(x)\sec^4(x)\,dx = \frac{1}{4}\tan^4(x) + \frac{1}{6}\tan^6(x) + C.$$

(c) We use the trigonometric substitution $x = \sqrt{\frac{2}{3}}\tan(\theta)$, $dx = \sqrt{\frac{2}{3}}\sec^2(\theta)\,d\theta$. Making the proper substitutions into the given integral we find that

$$
\begin{aligned}
\int \frac{1}{(2+3x^2)^{3/2}}\,dx &= \int \frac{1}{(2+3\cdot\frac{2}{3}\tan^2(\theta))^{3/2}}\sqrt{\frac{2}{3}}\sec^2(\theta)\,d\theta,\\
\int \frac{1}{(2+3x^2)^{3/2}}\,dx &= \sqrt{\frac{2}{3}}\int \frac{\sec^2(\theta)}{(2+2\tan^2(\theta))^{3/2}}\,d\theta,\\
\int \frac{1}{(2+3x^2)^{3/2}}\,dx &= \sqrt{\frac{2}{3}}\int \frac{\sec^2(\theta)}{(2(1+\tan^2(\theta)))^{3/2}}\,d\theta,\\
\int \frac{1}{(2+3x^2)^{3/2}}\,dx &= \sqrt{\frac{2}{3}}\int \frac{\sec^2(\theta)}{2^{3/2}(1+\tan^2(\theta))^{3/2}}\,d\theta,\\
\int \frac{1}{(2+3x^2)^{3/2}}\,dx &= \sqrt{\frac{2}{3}}\cdot\frac{1}{2^{3/2}}\int \frac{\sec^2(\theta)}{(\sec^2(\theta))^{3/2}}\,d\theta,\\
\int \frac{1}{(2+3x^2)^{3/2}}\,dx &= \sqrt{\frac{2}{3}}\cdot\frac{1}{2\sqrt{2}}\int \frac{\sec^2(\theta)}{\sec^3(\theta)}\,d\theta,\\
\int \frac{1}{(2+3x^2)^{3/2}}\,dx &= \frac{\sqrt{2}}{\sqrt{3}}\cdot\frac{1}{2\sqrt{2}}\int \cos(\theta)\,d\theta,\\
\int \frac{1}{(2+3x^2)^{3/2}}\,dx &= \frac{1}{2\sqrt{3}}\sin(\theta)+C.
\end{aligned}
$$

We use the fact that $x = \sqrt{\frac{2}{3}}\tan(\theta)$ and the identity $\sec^2(\theta) = 1+\tan^2(\theta)$ to find that

$$
\begin{aligned}
\sec^2(\theta) &= 1+\tan^2(\theta),\\
\sec^2(\theta) &= 1+\left(\frac{x}{\sqrt{\frac{2}{3}}}\right)^2,\\
\sec^2(\theta) &= 1+\frac{x^2}{\frac{2}{3}},\\
\sec^2(\theta) &= 1+\frac{3x^2}{2},\\
\sec(\theta) &= \sqrt{1+\frac{3x^2}{2}},\\
\sec(\theta) &= \sqrt{\frac{2+3x^2}{2}},\\
\cos(\theta) &= \sqrt{\frac{2}{2+3x^2}}.
\end{aligned}
$$

Finally, we use the identity $\sin^2(\theta) = 1 - \cos^2(\theta)$ to get

$$\begin{aligned}
\sin^2(\theta) &= 1 - \cos^2(\theta), \\
\sin^2(\theta) &= 1 - \left(\sqrt{\frac{2}{2+3x^2}}\right)^2, \\
\sin^2(\theta) &= 1 - \frac{2}{2+3x^2}, \\
\sin^2(\theta) &= \frac{3x^2}{2+3x^2}, \\
\sin(\theta) &= \sqrt{\frac{3x^2}{2+3x^2}}, \\
\sin(\theta) &= \frac{\sqrt{3}x}{\sqrt{2+3x^2}}.
\end{aligned}$$

The integral is then

$$\begin{aligned}
\int \frac{1}{(2+3x^2)^{3/2}}\,dx &= \frac{1}{2\sqrt{3}}\sin(\theta) + C, \\
\int \frac{1}{(2+3x^2)^{3/2}}\,dx &= \frac{1}{2\sqrt{3}} \cdot \frac{\sqrt{3}x}{\sqrt{2+3x^2}} + C, \\
\int \frac{1}{(2+3x^2)^{3/2}}\,dx &= \frac{x}{2\sqrt{2+3x^2}} + C.
\end{aligned}$$

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Problem 5 Solution

5. Calculate the definite integral: $\displaystyle\int_0^2 \frac{1}{(x+3)(x+5)}\,dx$.

Solution: The partial fraction decomposition of the integrand is

$$\frac{1}{(x+3)(x+5)} = \frac{A}{x+3} + \frac{B}{x+5}$$

After multiplying both sides of the equation by the denominator on the left hand side of the above equation we get

$$1 = A(x+5) + B(x+3)$$

Letting $x = -5$ we find that $B = -\frac{1}{2}$. Letting $x = -3$ we find that $A = \frac{1}{2}$. Therefore, the final decomposition is

$$\frac{1}{(x+3)(x+5)} = \frac{\frac{1}{2}}{x+3} - \frac{\frac{1}{2}}{x+5}$$

The value of the given integral is then

$$\int_0^2 \frac{1}{(x+3)(x+5)}\,dx = \int_0^2 \left(\frac{\frac{1}{2}}{x+3} - \frac{\frac{1}{2}}{x+5}\right)\,dx,$$

$$\int_0^2 \frac{1}{(x+3)(x+5)}\,dx = \left[\frac{1}{2}\ln(x+3) - \frac{1}{2}\ln(x+5)\right]_0^2,$$

$$\int_0^2 \frac{1}{(x+3)(x+5)}\,dx = \left[\frac{1}{2}\ln(2+3) - \frac{1}{2}\ln(2+5)\right] - \left[\frac{1}{2}\ln(0+3) - \frac{1}{2}\ln(0+5)\right],$$

$$\int_0^2 \frac{1}{(x+3)(x+5)}\,dx = \frac{1}{2}\ln(5) - \frac{1}{2}\ln(7) - \frac{1}{2}\ln(3) + \frac{1}{2}\ln(5),$$

$$\int_0^2 \frac{1}{(x+3)(x+5)}\,dx = \ln(5) - \frac{1}{2}\ln(7) - \frac{1}{2}\ln(3),$$

$$\int_0^2 \frac{1}{(x+3)(x+5)}\,dx = \ln\left(\frac{5}{\sqrt{21}}\right).$$