Math 181, Exam 1, Spring 2013
Problem 1 Solution

1. Compute the integrals $\int xe^{4x}\,dx$ and $\int \arctan x\,dx$.

Solution: We compute the first integral using Integration by Parts. The following table summarizes the elements that make up the technique.

$u = x$	$dv = e^{4x}\,dx$
$du = dx$	$v = \frac{1}{4}e^{4x}$

Using the Integration by Parts formula we have

$$\int u\,dv = uv - \int v\,du$$

$$\int xe^{4x}\,dx = \frac{1}{4}xe^{4x} - \frac{1}{4}\int e^{4x}\,dx$$

ANSWER $$\boxed{\int xe^{4x}\,dx = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + C}$$

The second integral is computed by Integration by Parts as well. The following table summaries the elements that make up the technique.

$u = \arctan(x)$	$dv = dx$
$du = \dfrac{1}{x^2+1}\,dx$	$v = x$

Using the Integration by Parts formula we have

$$\int u\,dv = uv - \int v\,du$$

$$\int \arctan(x)\,dx = x\arctan(x) - \int \frac{x}{x^2+1}\,dx$$

ANSWER $$\boxed{\int \arctan(x)\,dx = x\arctan(x) - \frac{1}{2}\ln(x^2+1) + C}$$

Math 181, Exam 1, Spring 2013
Problem 2 Solution

2. Compute the integral $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+4}}$.

Solution: We compute the integral using the trigonometric substitution $x = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$. After substituting these expressions into the integral and simplifying we obtain:

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{x^2+4}} &= \int \frac{2\sec^2\theta\,d\theta}{(2\tan\theta)^2\sqrt{(2\tan\theta)^2+4}} \\
&= \int \frac{2\sec^2\theta}{4\tan^2\theta\sqrt{4\tan^2\theta+4}}\,d\theta \\
&= \int \frac{2\sec^2\theta}{4\tan^2\theta\sqrt{4(\tan^2\theta+1)}}\,d\theta \\
&= \int \frac{2\sec^2\theta}{4\tan^2\theta\sqrt{4\sec^2\theta}}\,d\theta \\
&= \int \frac{2\sec^2\theta}{4\tan^2\theta\cdot 2\sec\theta}\,d\theta \\
&= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta
\end{aligned}$$

To evaluate the resulting trigonometric integral we rewrite the integrand in terms of $\sin\theta$ and $\cos\theta$ by using the definitions

$$\sec\theta = \frac{1}{\cos\theta}, \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

The integral then transforms into

$$\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

We can either

(1) rewrite the integrand as $\cot\theta\csc\theta$ and use the fact that this is the derivative of $-\csc\theta$ or

(2) use the substitution $u = \sin\theta$, $du = \cos\theta\,d\theta$.

In either case we obtain the result:

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = -\csc\theta + C$$

Therefore, our integral takes the form

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = -\frac{1}{4}\csc\theta + C$$

We must finish the problem by writing $\csc\theta$ in terms of x. Since we know that $x = 2\tan\theta$ we have

$$\tan\theta = \frac{x}{2} = \frac{\text{OPP}}{\text{ADJ}}$$

where OPP and ADJ are the opposite and adjacent sides of a right triangle, respectively, where opposite refers to the length of the side across from θ. Using the Pythagorean Theorem, the hypotenuse of this triangle is $\sqrt{x^2+4}$. Therefore,

$$\csc\theta = \frac{1}{\sin\theta} = \frac{\text{HYP}}{\text{OPP}} = \frac{\sqrt{x^2+4}}{x}$$

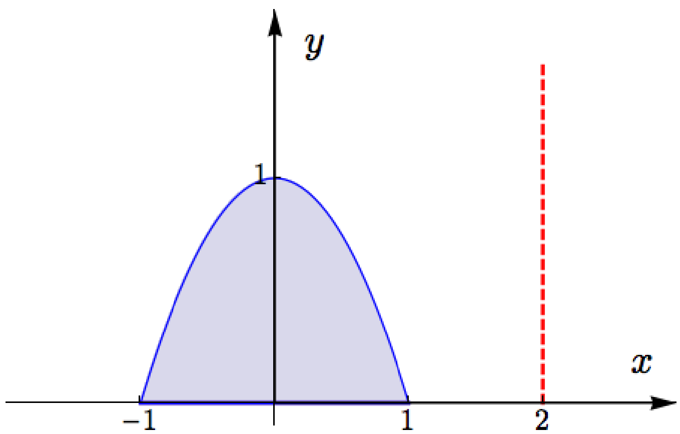
After substituting this expression into our result we find that

Answer $$\boxed{\int \frac{dx}{x^2\sqrt{x^2+4}} = -\frac{\sqrt{x^2+4}}{4x} + C}$$

Math 181, Exam 1, Spring 2013
Problem 3 Solution

3. The region between the x-axis and the parabola $y = 1 - x^2$ is rotated about the line $x = 2$. Find the volume of the resulting solid.

Solution: The region being rotated is plotted below.



In this case, we use the Shell Method because the integration with respect to x is easier to perform. The volume formula is

$$V = 2\pi \int_a^b (\text{radius})(\text{height})\,dx$$

The height of each shell is given by $1 - x^2$. Since the region is being rotated about the axis $x = 2$, the radius of each shell is given by $2 - x$. The interval over which the integral will take place is $x = -1$ to $x = 1$ since these are the points where the parabola $y = 1 - x^2$ intersects the x-axis. Therefore, the volume of the solid is

$$V = 2\pi \int_{-1}^{1} (2 - x)(1 - x^2)\,dx$$

$$V = 2\pi \int_{-1}^{1} (2 - x - 2x^2 + x^3)\,dx$$

$$V = 8\pi \int_{0}^{1} (1 - x^2)\,dx$$

$$V = 8\pi \left[x - \frac{x^3}{3}\right]_0^1$$

ANSWER $\boxed{V = \frac{16\pi}{3}}$

Math 181, Exam 1, Spring 2013
Problem 4 Solution

4. Compute the integrals $\displaystyle\int \frac{dx}{x^2 - x}$ and $\displaystyle\int \frac{x+3}{x^2+2x+5}\,dx$.

Solution: The integrand of $\displaystyle\int \frac{dx}{x^2 - x}$ is a rational function whose denominator factors into $x(x-1)$. Thus, we will use the method of partial fractions. The partial fraction decomposition of the integrand is

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

After clearing denominators we find that

$$1 = A(x-1) + Bx$$

When $x = 0$ we have $A = -1$ and when $x = 1$ we have $B = 1$. Therefore, we may evaluate the integral as follows:

$$\int \frac{dx}{x^2 - x} = \int \left(-\frac{1}{x} + \frac{1}{x-1} \right) dx$$

ANSWER $$\boxed{\int \frac{dx}{x^2 - x} = -\ln|x| + \ln|x-1| + C}$$

The integrand of the integral $\displaystyle\int \frac{x+3}{x^2+2x+5}\,dx$ is a rational function but the denominator is an irreducible quadratic. Therefore we begin by completing the square:

$$x^2 + 2x + 5 = (x^2 + 2x + 1) + 5 - 1 = (x+1)^2 + 4$$

At the same time we can rewrite the numerator as $x + 3 = (x+1) + 2$. The integral can then be split into the sum of two integrals

$$\int \frac{x+3}{x^2+2x+5}\,dx = \int \frac{x+1}{(x+1)^2+4}\,dx + \int \frac{2}{(x+1)^2+4}\,dx$$

Letting $u = x + 1$, $du = dx$ we obtain:

$$\int \frac{x+3}{x^2+2x+5}\,dx = \int \frac{u}{u^2+4}\,du + \int \frac{2}{u^2+4}\,du$$

The first integral on the right hand side may be evaluated using the substitution $v = u^2 + 4$, $\frac{1}{2}\,dv = u\,du$ and the second integral may be evaluated using the trigonometric substitution

$u = 2\tan\theta$, $du = 2\sec^2\theta\, d\theta$. The sum of these integrals transforms and evaluates as follows:

$$\begin{aligned}\int \frac{x+3}{x^2+2x+5}\,dx &= \int \frac{u}{u^2+4} + \int \frac{2}{u^2+4}\,du \\ &= \frac{1}{2}\int \frac{dv}{v} + \int d\theta \\ &= \frac{1}{2}\ln|v| + \theta + C \\ &= \frac{1}{2}\ln(u^2+4) + \arctan\left(\frac{u}{2}\right) + C\end{aligned}$$

where we used the fact that $v = u^2 + 4$ and $\theta = \arctan(\frac{u}{2})$ to write our answer in terms of u. We must take it a step further and write our answer in terms of x. We use the fact that $u = x + 1$ to obtain:

$$\int \frac{x+3}{x^2+2x+5}\,dx = \frac{1}{2}\ln((x+1)^2+4) + \arctan\left(\frac{x+1}{2}\right) + C$$

Answer $$\boxed{\int \frac{x+3}{x^2+2x+5}\,dx = \frac{1}{2}\ln(x^2+2x+5) + \arctan\left(\frac{x+1}{2}\right) + C}$$

Math 181, Exam 1, Spring 2013
Problem 5 Solution

5. Find the arc length of the graph of $f(x) = \ln x - \dfrac{x^2}{8}$ from 1 to e.

Solution: The arc length formula we will use is

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

The derivative $f'(x)$ is

$$f'(x) = \frac{1}{x} - \frac{x}{4}$$

Upon adding 1 to the square of $f'(x)$ we find that the result is a perfect square. The details are outlined below:

$$1 + f'(x)^2 = 1 + \left(\frac{1}{x} - \frac{x}{4}\right)^2$$
$$1 + f'(x)^2 = 1 + \frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{16}$$
$$1 + f'(x)^2 = \frac{1}{x^2} + \frac{1}{2} + \frac{x^2}{16}$$
$$1 + f'(x)^2 = \left(\frac{1}{x} + \frac{x}{4}\right)^2$$

Therefore, the arc length is

$$L = \int_1^e \sqrt{1 + f'(x)^2}\, dx$$
$$L = \int_1^e \left(\frac{1}{x} + \frac{x}{4}\right) dx$$
$$L = \left[\ln(x) + \frac{x^2}{8}\right]_1^e$$
$$L = \left[\ln(e) + \frac{e^2}{8}\right] - \left[\ln(1) + \frac{1^2}{8}\right]$$

ANSWER $\boxed{L = 1 + \frac{1}{8}(e^2 - 1)}$