Math 181, Exam 1, Study Guide
Problem 1 Solution

1. Evaluate $\int xe^{x^2}\,dx$.

Solution: We evaluate the integral using the u-substitution method. Let $u = x^2$. Then $du = 2x\,dx \quad \Rightarrow \quad \frac{1}{2}\,du = x\,dx$ and we get:

$$\begin{aligned}
\int xe^{x^2}\,dx &= \int e^{x^2} \cdot x\,dx \\
&= \int e^u \cdot \frac{1}{2}\,du \\
&= \frac{1}{2}\int e^u\,du \\
&= \frac{1}{2}e^u + C \\
&= \boxed{\frac{1}{2}e^{x^2} + C}
\end{aligned}$$

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Problem 2 Solution

2. Evaluate $\displaystyle\int \frac{1}{16x^2+1}\,dx$.

Solution: We evaluate the integral using the u-substitution method. Let $u = 4x$. Then $du = 4\,dx \quad \Rightarrow \quad \frac{1}{4}\,du = dx$ and we get:

$$\begin{aligned}
\int \frac{1}{16x^2+1}\,dx &= \int \frac{1}{(4x)^2+1}\,dx \\
&= \int \frac{1}{u^2+1}\cdot\frac{1}{4}\,du \\
&= \frac{1}{4}\int \frac{1}{u^2+1}\,du \\
&= \frac{1}{4}\arctan u + C \\
&= \boxed{\frac{1}{4}\arctan(4x) + C}
\end{aligned}$$

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Problem 3 Solution

3. Evaluate $\displaystyle\int \frac{t^3}{\sqrt{t^4+9}}\,dt$.

Solution: We evaluate the integral using the u-substitution method. Let $u = t^4 + 9$. Then $du = 4t^3\,dt \quad \Rightarrow \quad \dfrac{1}{4}\,du = t^3\,dt$ and we get:

$$\begin{aligned}
\int \frac{t^3}{\sqrt{t^4+9}}\,dt &= \int \frac{1}{\sqrt{t^4+9}} \cdot t^3\,dt \\
&= \int \frac{1}{\sqrt{u}} \cdot \frac{1}{4}\,du \\
&= \frac{1}{4}\int \frac{1}{\sqrt{u}}\,du \\
&= \frac{1}{4}\left[2\sqrt{u}\right] + C \\
&= \frac{1}{2}\sqrt{u} + C \\
&= \boxed{\frac{1}{2}\sqrt{t^4+9} + C}
\end{aligned}$$

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Problem 4 Solution

4. Evaluate $\int \tan^2\theta \sec^2\theta \, d\theta$.

Solution: We evaluate the integral using the u-substitution method. Let $u = \tan\theta$. Then $du = \sec^2\theta \, d\theta$ and we get:

$$\begin{aligned}\int \tan^2\theta \sec^2\theta \, d\theta &= \int u^2 \, du \\ &= \frac{1}{3}u^3 + C \\ &= \boxed{\frac{1}{3}\tan^3\theta + C}\end{aligned}$$

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Problem 5 Solution

5. Evaluate $\displaystyle\int_0^1 x\sqrt{1-x^2}\,dx$.

Solution: We evaluate the integral using the u-substitution method. Let $u = 1 - x^2$. Then $du = -2x\,dx \quad\Rightarrow\quad -\frac{1}{2}\,du = x\,dx$. The limits of integration becomes $u = 1 - 0^2 = 1$ and $u = 1 - 1^2 = 0$. We get:

$$\begin{aligned}
\int_0^1 x\sqrt{1-x^2}\,dx &= \int_0^1 \sqrt{1-x^2}\,(x\,dx) \\
&= \int_1^0 \sqrt{u}\left(-\frac{1}{2}\,du\right) \\
&= -\frac{1}{2}\int_1^0 u^{1/2}\,du \\
&= \frac{1}{2}\int_0^1 u^{1/2}\,du \\
&= \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_0^1 \\
&= \frac{1}{2}\left[\frac{2}{3}(1)^{3/2}\right] - \frac{1}{2}\left[\frac{2}{3}(0)^{3/2}\right] \\
&= \boxed{\frac{1}{3}}
\end{aligned}$$

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Problem 6 Solution

6. Evaluate $\displaystyle\int_{-\pi}^{\pi} \frac{\cos x}{\sqrt{4+3\sin x}}\,dx$.

Solution: We evaluate the integral using the u-substitution method. Let $u = 4 + 3\sin x$. Then $du = 3\cos x\,dx \quad \Rightarrow \quad \frac{1}{3}\,du = \cos x\,dx$. The limits of integration become $u = 4 + 3\sin(-\pi) = 4$ and $u = 4 + 3\sin\pi = 4$. We get:

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{\cos x}{\sqrt{4+3\sin x}}\,dx &= \int_{-\pi}^{\pi} \frac{1}{\sqrt{4+3\sin x}} \cdot \cos x\,dx \\
&= \int_{4}^{4} \frac{1}{\sqrt{u}} \cdot \frac{1}{3}\,du \\
&= \boxed{0}
\end{aligned}$$

because the limits of integration are identical.

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Problem 7 Solution

7. Evaluate $\displaystyle\int_0^1 \frac{4}{\sqrt{1-x^2}}\,dx$.

Solution: The integrand is undefined at $x = 1$. Therefore, this is an improper integral. We evaluate the integral by turning it into a limit calculation.

$$\begin{aligned}
\int_0^1 \frac{4}{\sqrt{1-x^2}}\,dx &= \lim_{b\to 1^-} \int_0^b \frac{4}{\sqrt{1-x^2}}\,dx \\
&= \lim_{b\to 1^-} \Big[4\arcsin x\Big]_0^b \\
&= \lim_{b\to 1^-} [4\arcsin b - 4\arcsin 0] \\
&= 4\arcsin 1 - 4\arcsin 0 \\
&= 4\left(\frac{\pi}{2}\right) - 4(0) \\
&= \boxed{2\pi}
\end{aligned}$$

We evaluated the limit $\displaystyle\lim_{b\to 1^-} 4\arcsin b$ by substituting $b = 1$ using the fact that $f(b) = 4\arcsin b$ is left-continuous at $b = 1$.

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Problem 8 Solution

8. Find $\frac{dy}{dx}$ for each of the following:

(a) $y = \int_0^x \sqrt{1+t^2}\,dt$

(b) $y = \int_0^{\sqrt{x}} \sin\left(t^2\right)\,dt$

(c) $y = \int_0^{\tan x} \frac{1}{1+t^2}\,dt$ (hint: when you simplify, you will get a constant)

Solution: In all parts, we use the Fundamental Theorem of Calculus Part II:

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))g'(x)$$

(a) The derivative is:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\int_0^x \sqrt{1+t^2}\,dt\\ &= \sqrt{1+(x)^2}\cdot(x)'\\ &= \boxed{\sqrt{1+x^2}}\end{aligned}$$

(b) The derivative is:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\int_0^{\sqrt{x}} \sin\left(t^2\right)\,dt\\ &= \sin\left(\left(\sqrt{x}\right)^2\right)\cdot\left(\sqrt{x}\right)'\\ &= \boxed{\sin x\cdot\frac{1}{2\sqrt{x}}}\end{aligned}$$

(c) The derivative is:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\int_0^{\tan x} \frac{1}{1+t^2}\,dt\\ &= \frac{1}{1+(\tan x)^2}\cdot(\tan x)'\\ &= \frac{1}{1+\tan^2 x}\cdot\sec^2 x\\ &= \frac{1}{\sec^2 x}\cdot\sec^2 x\\ &= \boxed{1}\end{aligned}$$

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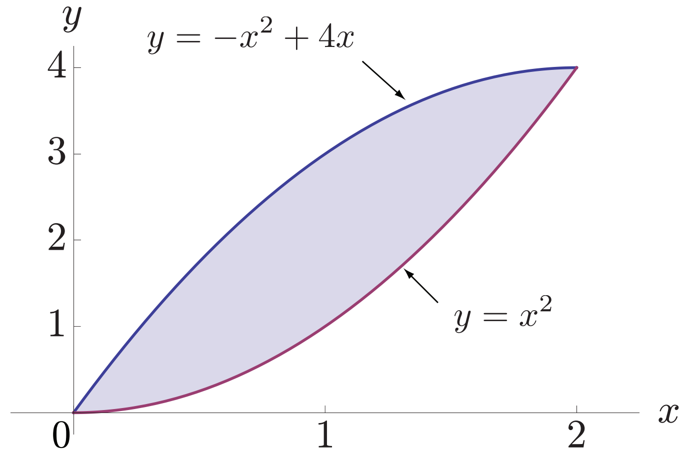
Problem 9 Solution

9. Find the area of the region enclosed by the curves $y = x^2$ and $y = -x^2 + 4x$.

Solution:



The formula we will use to compute the area of the region is:

$$\text{Area} = \int_a^b (\text{top} - \text{bottom})\, dx$$

where the limits of integration are the x-coordinates of the points of intersection of the two curves. These are found by setting the y's equal to each other and solving for x.

$$\begin{aligned} y &= y \\ x^2 &= -x^2 + 4x \\ 2x^2 - 4x &= 0 \\ 2x(x-2) &= 0 \\ x = 0,\ x &= 2 \end{aligned}$$

From the graph we see that the top curve is $y = -x^2 + 4x$ and the bottom curve is $y = x^2$.

Therefore, the area is:

$$\begin{aligned}
\text{Area} &= \int_a^b (\text{top} - \text{bottom})\ dx \\
&= \int_0^2 \left[\left(-x^2 + 4x\right) - \left(x^2\right)\right]\ dx \\
&= \int_0^2 \left(4x - 2x^2\right)\ dx \\
&= \left[2x^2 - \frac{2}{3}x^3\right]_0^2 \\
&= 2(2)^2 - \frac{2}{3}(2)^3 \\
&= \boxed{\frac{8}{3}}
\end{aligned}$$

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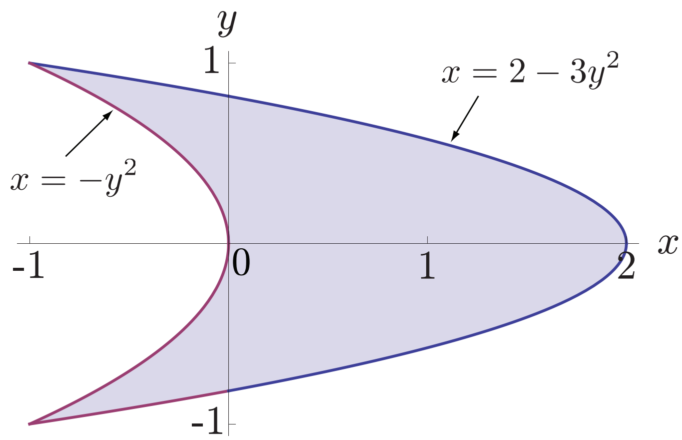
Problem 10 Solution

10. Find the area of the region enclosed by the curves $x + y^2 = 0$ and $x + 3y^2 = 2$.

Solution:



The formula we will use to compute the area of the region is:

$$\text{Area} = \int_c^d (\text{right} - \text{left})\ dx$$

where the limits of integration are the y-coordinates of the points of intersection of the two curves. These are found by setting the x's equal to each other and solving for y.

$$\begin{aligned} x &= x \\ -y^2 &= -3y^2 + 2 \\ 2y^2 - 2 &= 0 \\ 2(y^2 - 1) &= 0 \\ 2(y+1)(y-1) &= 0 \\ y = -1,\ y &= 1 \end{aligned}$$

From the graph we see that the right curve is $x = -3y^2 + 2$ and the left curve is $x = -y^2$.

Therefore, the area is:

$$\begin{aligned}
\text{Area} &= \int_c^d (\text{right} - \text{left})\ dx \\
&= \int_{-1}^{1} \left[\left(-3y^2 + 2\right) - \left(-y^2\right)\right]\ dy \\
&= \int_{-1}^{1} \left(2 - 2y^2\right)\ dy \\
&= \left[2y - \frac{2}{3}y^3\right]_{-1}^{1} \\
&= \left[2(1) - \frac{2}{3}(1)^3\right] - \left[2(-1) - \frac{2}{3}(-1)^3\right] \\
&= \boxed{\frac{8}{3}}
\end{aligned}$$

Math 181, Exam 1, Study Guide
Problem 11 Solution

11. Find the volume of the following solid:

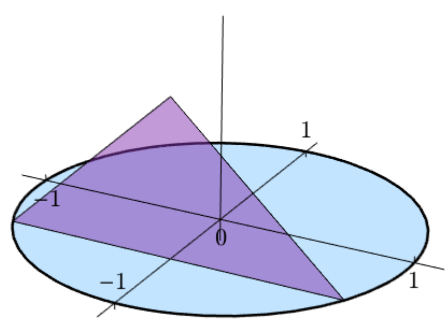
The base of the solid is the circle $x^2 + y^2 = 1$. The cross-sections are isosceles right triangles perpendicular to the y-axis.

Solution: The volume formula we will use is:

$$V = \int_c^d A(y)\,dy$$

where $A(y)$ is the cross sectional area of the solid as a function of y. We choose to integrate with respect to y in this problem because the cross sections are perpendicular to the y-axis.

To determine the function $A(y)$ we first note that the cross sections are isosceles right triangles. The triangle may be oriented in one of two ways: (1) with the hypotenuse lying inside the circle $x^2 + y^2 = 1$ or (2) with one of the sides lying inside the circle.



Let's assume (1) for the moment. In this case, the length of the hypotenuse is $2x$ where $x = \sqrt{1 - y^2}$ after solving the equation $x^2 + y^2 = 1$ for x. The other two sides of the triangle are equal since the triangle is isosceles. Letting the other sides be a, we use the Pythagorean Theorem to find a.

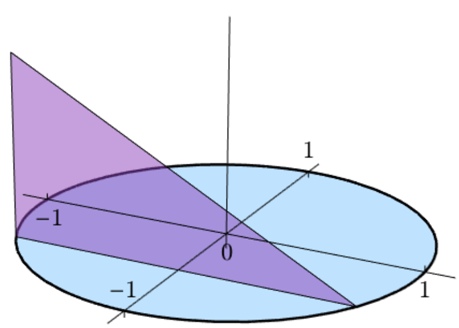
$$\begin{aligned} a^2 + a^2 &= (2x)^2 \\ 2a^2 &= 4x^2 \\ 2a^2 &= 4(1 - y^2) \\ a^2 &= 2(1 - y^2) \\ a &= \sqrt{2(1 - y^2)} \end{aligned}$$

The area of the triangle is:

$$\begin{aligned}A(y) &= \frac{1}{2}bh \\ &= \frac{1}{2}(a)(a) \\ &= \frac{1}{2}a^2 \\ &= \frac{1}{2}\left[2\left(1-y^2\right)\right] \\ &= 1-y^2\end{aligned}$$

The cross sections start at $c=-1$ and end at $d=1$. Therefore, the volume is:

$$\begin{aligned}V &= \int_{-1}^{1}\left(1-y^2\right)\,dy \\ &= \left[y-\frac{1}{3}y^3\right]_{-1}^{1} \\ &= \left[1-\frac{1}{3}(1)^3\right]-\left[(-1)-\frac{1}{3}(-1)^3\right] \\ &= 1-\frac{1}{3}+1-\frac{1}{3} \\ &= \boxed{\frac{4}{3}}\end{aligned}$$



Now let's assume (2), that one of the sides lies in the circle. In this case, the length of the side is $2x$ where $x=\sqrt{1-y^2}$. The other side of the triangle is also $2x$ since the triangle is isosceles. The area of the triangle is then:

$$\begin{aligned} A(y) &= \frac{1}{2}bh \\ &= \frac{1}{2}(2x)(2x) \\ &= 2x^2 \\ &= 2(1-y^2) \end{aligned}$$

The cross sections start at $c = -1$ and end at $d = 1$. Therefore, the volume is:

$$\begin{aligned} V &= \int_{-1}^{1} 2\left(1-y^2\right)\, dy \\ &= 2\left[y - \frac{1}{3}y^3\right]_{-1}^{1} \\ &= 2\left[1 - \frac{1}{3}(1)^3\right] - \left[(-1) - \frac{1}{3}(-1)^3\right] \\ &= 2\left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) \\ &= \boxed{\frac{8}{3}} \end{aligned}$$

Math 181, Exam 1, Study Guide
Problem 12 Solution

12. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$, $y = 2$, and $x = 0$ about the:

(a) x-axis

(b) y-axis

Solution:

(a) We find the volume of the solid generated by revolving around the x-axis using the **Washer Method**. The variable of integration is x and the corresponding formula is:

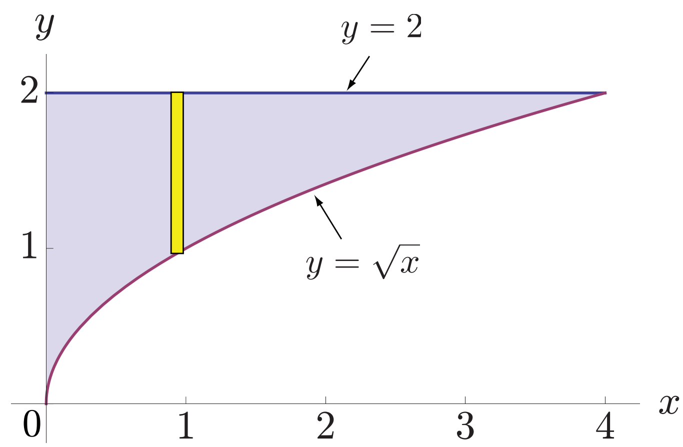
$$V = \pi \int_a^b \left[(\text{top})^2 - (\text{bottom})^2\right] \, dx$$

The top curve is $y = 2$ and the bottom curve is $y = \sqrt{x}$. The lower limit of integration is $x = 0$. The upper limit is the x-coordinate of the point of intersection of the curves $y = 2$ and $y = \sqrt{x}$. To find this, we set the y's equal to each other and solve for x.

$$\begin{aligned} y &= y \\ \sqrt{x} &= 2 \\ x &= 4 \end{aligned}$$

The volume is then:

$$\begin{aligned} V &= \pi \int_0^4 \left[(2)^2 - \left(\sqrt{x}\right)^2\right] \, dx \\ &= \pi \int_0^4 (4 - x) \, dx \\ &= \pi \left[4x - \frac{1}{2}x^2\right]_0^4 \\ &= \pi \left[4(4) - \frac{1}{2}(4)^2\right] \\ &= \boxed{8\pi} \end{aligned}$$



(b) We find the volume of the solid generated by revolving around the y-axis using the **Shell Method**. The variable of integration is x and the corresponding formula is:

$$V = 2\pi \int_a^b x\,(\text{top} - \text{bottom})\;dx$$

The top and bottom curves are the same as those in part (a). So are the limits of integration. The volume is then:

$$\begin{aligned} V &= 2\pi \int_0^4 x\left(2 - \sqrt{x}\right)\;dx \\ &= 2\pi \int_0^4 \left(2x - x^{3/2}\right)\;dx \\ &= 2\pi \left[x^2 - \frac{2}{5}x^{5/2}\right]_0^4 \\ &= 2\pi \left[4^2 - \frac{2}{5}(4)^{5/2}\right] \\ &= \boxed{\frac{32\pi}{5}} \end{aligned}$$

Math 181, Exam 1, Study Guide
Problem 13 Solution

13. Find the volume of the solid generated by revolving the region bounded by the y-axis and the curve $x = \dfrac{2}{y}$, for $1 \leq y \leq 4$, about the:

(a) x-axis, using the Method of Cylindrical Shells

(b) y-axis

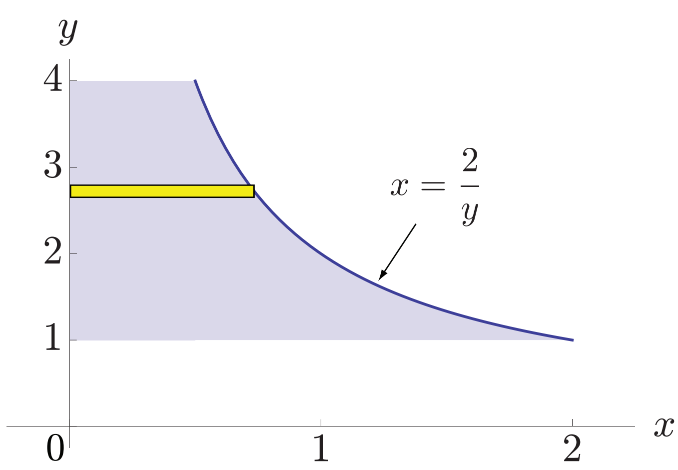
Solution:

(a) We find the volume of the solid obtained by rotating about the x-axis using the **Shell Method**. The variable of integration is y and the corresponding formula is:

$$V = 2\pi \int_c^d y\,(\text{right} - \text{left})\; dy$$

The right curve is $x = \dfrac{2}{y}$ and the left curve is $x = 0$ (the y-axis). The volume is then:

$$\begin{aligned} V &= 2\pi \int_1^4 y\left(\frac{2}{y} - 0\right)\; dy \\ &= 2\pi \int_1^4 2\,dy \\ &= 2\pi \Big[2y\Big]_1^4 \\ &= 2\pi\,[2(4) - 2(1)] \\ &= \boxed{12\pi} \end{aligned}$$



(b) We find the volume of the solid obtained by rotating about the y-axis using the **Disk Method**. The variable of integration is y and the corresponding formula is:

$$V = \pi \int_c^d f(y)^2 \, dy$$

where $f(y) = \dfrac{2}{y}$. The volume is then:

$$\begin{aligned} V &= \pi \int_1^4 \left(\frac{2}{y}\right)^2 \, dy \\ &= \pi \int_1^4 4y^{-2} \, dy \\ &= 4\pi \left[-\frac{1}{y} \right]_1^4 \\ &= 4\pi \left[\left(-\frac{1}{4}\right) - \left(-\frac{1}{1}\right) \right] \\ &= \boxed{3\pi} \end{aligned}$$

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Problem 14 Solution

14. In some chemical reactions, the rate at which the amount of a substance changes with time is proportional to the amount present. Consider a substance whose amount obeys the equation:

$$\frac{dy}{dt} = -0.6y$$

where t is measured in hours. If there are 100 grams of the substance present when $t = 0$, how many grams will be left after 1 hour?

Solution: The amount of the substance $y(t)$ is given by the formula:

$$y(t) = y_0 e^{-0.06t}$$

where $y_0 = 100$ grams is the initial amount of the substance. After 1 hour, the amount of the substance is:

$$\boxed{y(1) = 100e^{-0.06} \text{ grams}}$$

Math 181, Exam 1, Study Guide
Problem 15 Solution

15. Suppose the rate at which the number of people infected with a disease $\frac{dy}{dt}$ is proportional to the number of people currently infected y:

$$\frac{dy}{dt} = ky$$

Suppose that, in the course of any given year, the number of people infected is reduced by 20%. If there are 10,000 infected people today, how many years will it take to reduce the number to 1000?

Solution: The number of people infected is given by the function:

$$y(t) = y_0 e^{kt}$$

where $y_0 = 10,000$ is the initial number of people infected. To answer the question in the problem, we need to find the value of k. Since the number of people infected is reduced by 20% in the course of any given year, the number of people infected after the first year is:

$$10,000 - 0.20(10,000) = 8,000$$

This corresponds to the value $y(1)$. Using the function above for $y(t)$ we get:

$$\begin{aligned}
y(1) &= 10,000e^{k(1)} \\
8,000 &= 10,000e^{k} \\
e^k &= \frac{8,000}{10,000} \\
e^k &= \frac{4}{5} \\
k &= \ln\frac{4}{5}
\end{aligned}$$

To find how many years it will take for the number of infected people to reduce to 1,000, we set $y(t)$ equal to 1,000 and solve for t.

$$\begin{aligned}
y(t) &= 10,000e^{(\ln\frac{4}{5})t} \\
1,000 &= 10,000e^{(\ln\frac{4}{5})t} \\
\frac{1,000}{10,000} &= e^{(\ln\frac{4}{5})t} \\
\frac{1}{10} &= e^{(\ln\frac{4}{5})t} \\
\ln\frac{1}{10} &= \left(\ln\frac{4}{5}\right)t \\
&\boxed{t = \frac{\ln\frac{1}{10}}{\ln\frac{4}{5}}}
\end{aligned}$$

Math 181, Exam 1, Study Guide
Problem 16 Solution

16. Consider the definite integral:

$$\int_0^4 \left(x^2 + x\right)\,dx$$

(a) Compute the exact value of the integral.

(b) Estimate the value of the integral using the Trapezoidal Rule with $N = 4$.

(c) Estimate the value of the integral using Simpson's Rule with $N = 4$.

(d) Which of the above two estimate is more accurate?

Solution:

(a) The exact value is:

$$\begin{aligned}\int_0^4 \left(x^2 + x\right)\,dx &= \left[\frac{1}{3}x^3 + \frac{1}{2}x^2\right]_0^4 \\ &= \frac{1}{3}(4)^3 + \frac{1}{2}(4)^2 \\ &= \boxed{\frac{88}{3}}\end{aligned}$$

(b) Using $N = 4$, the length of each subinterval of $[0, 4]$ is:

$$\Delta x = \frac{b-a}{N} = \frac{4-0}{4} = 1$$

The estimate T_4 is:

$$\begin{aligned}T_4 &= \frac{\Delta x}{2}\left[f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)\right] \\ &= \frac{1}{2}\left[(0^2+0) + 2(1^2+1) + 2(2^2+2) + 2(3^2+3) + (4^2+4)\right] \\ &= \frac{1}{2}\left[0 + 4 + 12 + 24 + 20\right] \\ &= \boxed{30}\end{aligned}$$

(c) The estimate S_4 is:

$$\begin{aligned} S_4 &= \frac{\Delta x}{3}\left[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)\right] \\ &= \frac{1}{3}\left[(0^2+0) + 4(1^2+1) + 2(2^2+2) + 4(3^2+3) + (4^2+4)\right] \\ &= \frac{1}{3}\left[0 + 8 + 12 + 48 + 20\right] \\ &= \boxed{\frac{88}{3}} \end{aligned}$$

(d) Clearly, S_4 is more accurate because it is the exact value of the integral.

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Problem 17 Solution

17. Consider the definite integral:

$$\int_0^2 \frac{dx}{1+x^2}$$

Estimate the value of the integral using:

(a) the Trapezoidal Rule with $N = 2$

(b) the Midpoint method with $N = 2$

(c) Simpson's Rule with $N = 4$

Solution:

(a) The length of each subinterval of $[0, 2]$ is:

$$\Delta x = \frac{b-a}{N} = \frac{2-0}{2} = 1$$

The estimate T_2 is:

$$\begin{aligned} T_2 &= \frac{\Delta x}{2}\left[f(0) + 2f(1) + f(2)\right] \\ &= \frac{1}{2}\left[\frac{1}{1+0^2} + 2\cdot\frac{1}{1+1^2} + \frac{1}{1+2^2}\right] \\ &= \frac{1}{2}\left[1 + 1 + \frac{1}{5}\right] \\ &= \boxed{\frac{11}{10}} \end{aligned}$$

(b) The length of each subinterval of $[0, 2]$ is $\Delta x = 1$ just as in part (a). The estimate M_2 is:

$$\begin{aligned} M_2 &= \Delta x\left[f\left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right)\right] \\ &= 1\cdot\left[\frac{1}{1+\left(\frac{1}{2}\right)^2} + \frac{1}{1+\left(\frac{3}{2}\right)^2}\right] \\ &= \frac{4}{5} + \frac{4}{13} \\ &= \boxed{\frac{72}{65}} \end{aligned}$$

(c) We can use the following formula to find S_4:

$$S_4 = \frac{2}{3}M_2 + \frac{1}{3}T_2$$

where M_2 and T_2 were found in parts (a) and (b). We get:

$$\begin{aligned} S_4 &= \frac{2}{3}\left(\frac{72}{65}\right) + \frac{1}{3}\left(\frac{11}{10}\right) \\ &= \frac{48}{65} + \frac{11}{30} \\ &= \boxed{\frac{431}{390}} \end{aligned}$$