Math 181, Exam 2, Fall 2007
Problem 1 Solution

1. Compute the integral:

$$\int 2x\sqrt{1-x^4}\,dx$$

Solution: We begin by using the u-substitution method. Let $u = x^2$. Then $du = 2x\,dx$ and we get:

$$\begin{aligned}\int 2x\sqrt{1-x^4}\,dx &= \int 2x\sqrt{1-(x^2)^2}\,dx \\ &= \int \sqrt{1-u^2}\,du\end{aligned}$$

We now use a trigonometric substitution to evaluate this integral. Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$ and we get:

$$\begin{aligned}\int \sqrt{1-u^2}\,du &= \int \sqrt{1-\sin^2\theta}\,(\cos\theta\,d\theta) \\ &= \int \cos\theta\cos\theta\,d\theta \\ &= \int \cos^2\theta\,d\theta \\ &= \int \frac{1}{2}[1+\cos(2\theta)]\,d\theta \\ &= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C \\ &= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C\end{aligned}$$

To write the result in terms of u we use the fact that:

$$\theta = \arcsin u, \qquad \sin\theta = u, \qquad \cos\theta = \sqrt{1-u^2}$$

to get:

$$\int \sqrt{1-u^2}\,du = \frac{1}{2}\arcsin u + \frac{1}{2}u\sqrt{1-u^2} + C$$

Finally, we write the answer in terms of x replacing u with x^2:

$$\boxed{\int 2x\sqrt{1-x^4}\,dx = \frac{1}{2}\arcsin\left(x^2\right) + \frac{1}{2}x^2\sqrt{1-x^4} + C}$$

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Problem 2 Solution

2. Determine whether the following integrals converge or not:

$$\int_2^{+\infty} \frac{x}{x^2+1}\,dx \qquad \int_0^1 \frac{dx}{x^{1/3}}$$

Solution: The first integral is improper due to the infinite upper limit of integration. We will evaluate the integral by turning it into a limit calculation.

$$\int_2^{+\infty} \frac{x}{x^2+1}\,dx = \lim_{R\to+\infty} \int_2^{R} \frac{x}{x^2+1}\,dx$$

We use the u-substitution method to compute the integral. Let $u = x^2+1$ and $du = 2x\,dx \quad \Rightarrow \quad \frac{1}{2}\,du = x\,dx$. The indefinite integral is then:

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln\left(x^2+1\right)$$

The definite integral from 2 to R is:

$$\begin{aligned}\int_2^{R} \frac{x}{x^2+1}\,dx &= \frac{1}{2}\Big[\ln\left(x^2+1\right)\Big]_2^R \\ &= \frac{1}{2}\Big[\ln\left(R^2+1\right) - \ln(2^2+1)\Big]\end{aligned}$$

Taking the limit as $R \to +\infty$ we get:

$$\begin{aligned}\int_2^{+\infty} \frac{x}{x^2+1}\,dx &= \lim_{R\to+\infty} \int_2^{R} \frac{x}{x^2+1}\,dx \\ &= \lim_{R\to+\infty} \frac{1}{2}\Big[\ln\left(R^2+1\right) - \ln\left(2^2+1\right)\Big] \\ &= \frac{1}{2}\Big[+\infty - \ln 5\Big] \\ &= \infty\end{aligned}$$

Therefore, the integral **diverges**.

The second integral is a p-integral of the form $\displaystyle\int_0^1 \frac{dx}{x^p}$ where $p = \frac{1}{3} < 1$. Therefore, the integral **converges** and its value is:

$$\int_0^1 \frac{dx}{x^{1/3}} = \frac{1}{1-\frac{1}{3}} = \boxed{\frac{3}{2}}$$

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Problem 3 Solution

3. Compute the following integral:

$$\int_{2}^{+\infty} xe^{-3x}\,dx$$

Solution: We evaluate the integral by turning it into a limit calculation.

$$\int_{2}^{+\infty} xe^{-3x}\,dx = \lim_{R\to+\infty} \int_{2}^{R} xe^{-3x}\,dx$$

We use Integration by Parts to compute the integral. Let $u = x$ and $v' = e^{-3x}$. Then $u' = 1$ and $v = -\frac{1}{3}e^{-x}$. Using the Integration by Parts formula we get:

$$\begin{aligned}
\int_{a}^{b} uv'\,dx &= \Big[uv\Big]_{a}^{b} - \int_{a}^{b} u'v\,dx \\
\int_{2}^{R} xe^{-3x}\,dx &= \left[-\frac{1}{3}xe^{-3x}\right]_{2}^{R} - \int_{2}^{R}\left(-\frac{1}{3}e^{-3x}\right)dx \\
&= \left[-\frac{1}{3}xe^{-3x}\right]_{2}^{R} + \frac{1}{3}\int_{2}^{R} e^{-3x}\,dx \\
&= \left[-\frac{1}{3}xe^{-3x}\right]_{2}^{R} + \frac{1}{3}\left[-\frac{1}{3}e^{-3x}\right]_{2}^{R} \\
&= \left[-\frac{1}{3}Re^{-3R} + \frac{1}{3}(2)e^{-3(2)}\right] + \frac{1}{3}\left[-\frac{1}{3}e^{-3R} + \frac{1}{3}e^{-3(2)}\right] \\
&= -\frac{R}{3e^{3R}} + \frac{2}{3e^{6}} - \frac{1}{9e^{3R}} + \frac{1}{9e^{6}}
\end{aligned}$$

We now take the limit of the above function as $R \to +\infty$.

$$\begin{aligned}
\int_2^{+\infty} xe^{-3x}\,dx &= \lim_{R\to+\infty} \int_2^R xe^{-3x}\,dx \\
&= \lim_{R\to+\infty} \left(-\frac{R}{3e^{3R}} + \frac{2}{3e^6} - \frac{1}{9e^{3R}} + \frac{1}{9e^6}\right) \\
&= -\lim_{R\to+\infty} \frac{R}{3e^{3R}} + \frac{2}{3e^6} - \lim_{R\to+\infty} \frac{1}{9e^{3R}} + \frac{1}{9e^6} \\
&= -\lim_{R\to+\infty} \frac{R}{3e^{3R}} + \frac{2}{3e^6} - 0 + \frac{1}{9e^6} \\
&\overset{\text{L'H}}{=} -\lim_{R\to+\infty} \frac{(R)'}{(3e^{3R})'} + \frac{2}{3e^6} - 0 + \frac{1}{9e^6} \\
&= -\lim_{R\to+\infty} \frac{1}{9e^{3R}} + \frac{2}{3e^6} - 0 + \frac{1}{9e^6} \\
&= -0 + \frac{2}{3e^6} - 0 + \frac{1}{9e^6} \\
&= \boxed{\frac{7}{9e^6}}
\end{aligned}$$

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Problem 4 Solution

4. Let R be the region defined by the x-axis, the graph of $y = 2x^4$, and the lines $x = 0$ and $x = 1$. Compute the volume of revolution obtained by rotating R around the x-axis.

Solution: We will use the **Disk Method** to compute the volume. The formula is:

$$V = \pi \int_a^b f(x)\, dx$$

where $f(x) = 2x^4$, $a = 0$, and $b = 1$. The volume is then:

$$\begin{aligned} V &= \pi \int_0^1 \left(2x^4\right)^2\, dx \\ &= \pi \int_0^1 4x^8\, dx \\ &= \pi \left[\frac{4}{9}x^9\right]_0^1 \\ &= \boxed{\frac{4\pi}{9}} \end{aligned}$$

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Problem 5 Solution

5. Find the sum of the series:

$$\sum_{n=3}^{+\infty} \frac{3^{2n+3}}{4^{3n-1}}$$

Solution: We recognize the given series as a geometric series. In order to find its sum we must first rewrite the series.

$$\sum_{n=3}^{+\infty} \frac{3^{2n+3}}{4^{3n-1}} = \sum_{n=3}^{+\infty} \frac{3^{2n}3^3}{4^{3n}4^{-1}} = \sum_{n=3}^{+\infty} \frac{3^3}{4^{-1}} \cdot \frac{(3^2)^n}{(4^3)^n} = \sum_{n=3}^{+\infty} 108 \left(\frac{9}{64}\right)^n$$

This is a convergent geometric series because $|r| = |\frac{9}{64}| < 1$. We can now use the formula:

$$\sum_{n=M}^{+\infty} cr^n = r^M \cdot \frac{c}{1-r}$$

where $M = 3$, $c = 108$, and $r = \frac{9}{64}$. The sum of the series is then:

$$\sum_{n=3}^{+\infty} 108 \left(\frac{9}{64}\right)^n = \left(\frac{9}{64}\right)^3 \cdot \frac{108}{1-\frac{9}{64}} = \boxed{\frac{19,683}{56,320}}$$

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Problem 6 Solution

6. Suppose that the random variable T has density function:

$$p(t) = \begin{cases} 5t^4 & \text{if } 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute the average of T and the median of T and the probability that T lies between 1/2 and 2.

Solution: The average of T is:

$$\int_a^b tp(t)\,dt = \int_0^1 t\left(5t^4\right)\,dt = \int_0^1 5t^5\,dt = \left[\frac{5}{6}t^6\right]_0^1 = \boxed{\frac{5}{6}}$$

The median of T is computed as follows:

$$\begin{aligned} \int_a^x p(t)\,dt &= \frac{1}{2} \\ \int_0^x 5t^4\,dt &= \frac{1}{2} \\ \left[t^5\right]_0^x &= \frac{1}{2} \\ x^5 &= \frac{1}{2} \\ x &= \boxed{\frac{1}{\sqrt[5]{2}}} \end{aligned}$$

The probability that T lies between 1/2 and 2 is:

$$\int_{1/2}^2 p(t)\,dt = \int_{1/2}^1 5t^4\,dt = \left[t^5\right]_{1/2}^1 = 1^5 - \left(\frac{1}{2}\right)^5 = \boxed{\frac{31}{32}}$$

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Problem 7 Solution

7. Compute the area enclosed by the curve $r = 3\theta^3$ and the two axes in the first quadrant.

Solution: The formula for the area of the region bounded by the polar curve $r = f(\theta)$ and the two rays $\theta = \alpha$ and $\theta = \beta$ is:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} f(\theta)^2 \, d\theta$$

For $f(\theta) = 3\theta^3$, $\alpha = 0$, and $\beta = \frac{\pi}{2}$ we have:

$$\begin{aligned} A &= \frac{1}{2}\int_0^{\pi/2} \left(3\theta^3\right)^2 \, d\theta \\ &= \frac{1}{2}\int_0^{\pi/2} 9\theta^6 \, d\theta \\ &= \frac{1}{2}\left[\frac{9}{7}\theta^7\right]_0^{\pi/2} \\ &= \boxed{\frac{9}{14}\left(\frac{\pi}{2}\right)^7} \end{aligned}$$

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Problem 8 Solution

8. Find the length of the graph of the function $f(x) = 6x^{3/2} + 1988$ between the points corresponding to $x = 0$ and $x = 1$.

Solution: The arclength is:

$$\begin{aligned} L &= \int_a^b \sqrt{1 + f'(x)^2}\, dx \\ &= \int_0^1 \sqrt{1 + \left(9x^{1/2}\right)^2}\, dx \\ &= \int_0^1 \sqrt{1 + 81x}\, dx \end{aligned}$$

We now use the u-substitution $u = 1 + 81x$. Then $\frac{1}{81}\, du = dx$, the lower limit of integration changes from 0 to 1, and the upper limit of integration changes from 1 to 82.

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + 81x}\, dx \\ &= \frac{1}{81} \int_1^{82} \sqrt{u}\, du \\ &= \frac{1}{81} \left[\frac{2}{3} u^{3/2} \right]_1^{82} \\ &= \frac{1}{81} \left[\frac{2}{3} (82)^{3/2} - \frac{2}{3} (1)^{3/2} \right] \\ &= \boxed{\frac{2}{243} \left[82^{3/2} - 1 \right]} \end{aligned}$$