Math 181, Exam 2, Fall 2012
Problem 1 Solution

1. Find the sums of the following series.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{6}{(k+2)(k+3)}$

(b) $\displaystyle\sum_{k=2}^{\infty} \frac{2^k+4}{e^k}$

Solution:

(a) This series is telescoping. We begin by decomposing the summand using partial fractions. The result is

$$\frac{6}{(k+2)(k+3)} = \frac{6}{k+2} - \frac{6}{k+3}$$

The nth partial sum of the series is:

$$S_n = 6\sum_{k=1}^{n}\left(\frac{1}{k+2} - \frac{1}{k+3}\right)$$
$$S_n = 6\left[\left(\frac{1}{3}-\frac{1}{4}\right) + \left(\frac{1}{4}-\frac{1}{5}\right) + \cdots + \left(\frac{1}{n+2}-\frac{1}{n+3}\right)\right]$$
$$S_n = 6\left[\frac{1}{3} - \frac{1}{n+3}\right]$$

The sum is then

$$\sum_{k=1}^{\infty} \frac{6}{(k+2)(k+3)} = \lim_{n\to\infty} S_n$$
$$\sum_{k=1}^{\infty} \frac{6}{(k+2)(k+3)} = \lim_{n\to\infty} 6\left[\frac{1}{3} - \frac{1}{n+3}\right]$$
$$\sum_{k=1}^{\infty} \frac{6}{(k+2)(k+3)} = 6\left[\frac{1}{3} - 0\right]$$
$$\sum_{k=1}^{\infty} \frac{6}{(k+2)(k+3)} = 2$$

(b) This is a sum of two geometric series. We begin by rewriting the series as follows:

$$\sum_{k=2}^{\infty} \frac{2^k+4}{e^k} = \sum_{k=2}^{\infty}\left(\frac{2}{e}\right)^k + \sum_{k=2}^{\infty} 4\left(\frac{1}{e}\right)^k$$

In the first series on the right hand side, we have $r = \frac{2}{e}$ and $a = 1$. Since the series starts at $k = 2$, the sum is

$$\sum_{k=2}^{\infty} \left(\frac{2}{e}\right)^k = \left(\frac{2}{e}\right)^2 \cdot \frac{1}{1 - \frac{2}{e}} = \frac{4}{e^2 - 2e}$$

In the second series on the right hand side, we have $r = \frac{1}{e}$ and $a = 4$. Since the series starts at $k = 2$, the sum is

$$\sum_{k=2}^{\infty} 4 \left(\frac{1}{e}\right)^k = \left(\frac{1}{e}\right)^2 \cdot \frac{4}{1 - \frac{1}{e}} = \frac{4}{e^2 - e}$$

Thus, the sum of the series is

$$\sum_{k=2}^{\infty} \frac{2^k + 4}{e^k} = \frac{4}{e^2 - 2e} + \frac{4}{e^2 - e}$$

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Problem 2 Solution

2. Evaluate each integral or show that it diverges.

(a) $\displaystyle\int_1^\infty \frac{x}{x^4+1}\,dx$

(b) $\displaystyle\int_0^1 \frac{2}{x(x+2)}\,dx$

Solution:

(a) The first step we take is to convert the integral into a limit calculation:

$$\int_1^\infty \frac{x}{x^4+1}\,dx = \lim_{b\to\infty}\int_1^b \frac{x}{x^4+1}\,dx$$

To evaluate the integral we make the substitution $u = x^2$, $\frac{1}{2}\,du = x\,dx$. Focusing on the indefinite integral we have

$$\int \frac{x}{x^4+1}\,dx = \frac{1}{2}\int \frac{1}{u^2+1}\,du = \frac{1}{2}\arctan(u) = \frac{1}{2}\arctan(x^2)$$

We now evaluate the improper integral as follows:

$$\int_1^\infty \frac{x}{x^4+1}\,dx = \lim_{b\to\infty}\int_1^b \frac{x}{x^4+1}\,dx$$

$$\int_1^\infty \frac{x}{x^4+1}\,dx = \lim_{b\to\infty}\left[\frac{1}{2}\arctan(x^2)\right]_1^b$$

$$\int_1^\infty \frac{x}{x^4+1}\,dx = \lim_{b\to\infty}\frac{1}{2}\arctan(b^2) - \frac{1}{2}\arctan(1)$$

$$\int_1^\infty \frac{x}{x^4+1}\,dx = \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\cdot\frac{\pi}{4}$$

$$\int_1^\infty \frac{x}{x^4+1}\,dx = \frac{\pi}{8}$$

(b) Again, the first step is to convert the integral into a limit calculation. Since the integrand has an infinite discontinuity at $x = 0$, we replace the lower limit of integration with a and take the limit as $a \to 0^+$:

$$\int_0^1 \frac{2}{x(x+2)}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{2}{x(x+2)}\,dx$$

The integrand decomposes into:

$$\frac{2}{x(x+2)} = \frac{1}{x} - \frac{1}{x+2}$$

by way of the method of partial fractions. We now evaluate the improper integral as follows:

$$\int_0^1 \frac{2}{x(x+2)}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{2}{x(x+2)}\,dx$$

$$\int_0^1 \frac{2}{x(x+2)}\,dx = \lim_{a\to 0^+} \Big[\ln(x) - \ln(x+2)\Big]_a^1$$

$$\int_0^1 \frac{2}{x(x+2)}\,dx = \lim_{a\to 0^+} \left[\ln\left(\frac{x}{x+2}\right)\right]_a^1$$

$$\int_0^1 \frac{2}{x(x+2)}\,dx = \lim_{a\to 0^+} \ln\left(\frac{1}{1+2}\right) - \ln\left(\frac{a}{a+2}\right)$$

$$\int_0^1 \frac{2}{x(x+2)}\,dx = \ln\left(\frac{1}{3}\right) - (-\infty)$$

$$\int_0^1 \frac{2}{x(x+2)}\,dx = \infty$$

Thus, the integral diverges.

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Problem 3 Solution

3. Find the limits of the following sequences or show that they diverge.

(a) $\left\{\dfrac{2n-\sin(n)}{4n+1}\right\}$

(b) $\left\{\dfrac{n2^n}{3^n}\right\}$

Solution:

(a) Since $-1 \leq \sin(n) \leq 1$ for all n we have

$$\frac{2n-1}{4n+1} \leq \frac{2n-\sin(n)}{4n+1} \leq \frac{2n+1}{4n+1}.$$

Moreover,

$$\lim_{n\to\infty} \frac{2n-1}{4n+1} = \frac{2}{4} = \frac{1}{2},$$
$$\lim_{n\to\infty} \frac{2n+1}{4n+1} = \frac{2}{4} = \frac{1}{2}.$$

Thus, $\displaystyle\lim_{n\to\infty} \frac{2n-\sin(n)}{4n+1} = \frac{1}{2}$ by the Squeeze Theorem.

(b) We begin be rewriting $f(n)$ as follows:

$$\frac{n2^n}{3^n} = \frac{n}{(\frac{3}{2})^n}.$$

Using the fact that $n \ll (\frac{3}{2})^n$ as $n \to \infty$ we have

$$\lim_{n\to\infty} \frac{n2^n}{3^n} = \lim_{n\to\infty} \frac{n}{(\frac{3}{2})^n} = 0.$$

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Problem 4 Solution

4. Approximate the value of the definite integral $\int_0^2 \frac{1}{2x+1}\,dx$ using

(a) the Midpoint Rule with $N = 2$ and

(b) the Trapezoidal Rule with $N = 2$.

Solution:

(a) Since $N = 2$ we have $\Delta x = \frac{b-a}{N} = \frac{2-0}{2} = 1$. The interval $[0, 2]$ is partitioned into the intervals $[0, 1]$ and $[1, 2]$. The midpoints of these intervals are $\frac{1}{2}$ and $\frac{3}{2}$. Thus, the Midpoint estimate of the integral is

$$M_2 = \Delta x \left[f(\tfrac{1}{2}) + f(\tfrac{3}{2})\right]$$
$$M_2 = 1 \cdot \left[\frac{1}{2(\frac{1}{2})+1} + \frac{1}{2(\frac{3}{2})+1}\right]$$
$$M_2 = 1 \cdot \left[\frac{1}{1+1} + \frac{1}{3+1}\right]$$
$$M_2 = 1 \cdot \left[\frac{1}{2} + \frac{1}{4}\right]$$
$$M_2 = \frac{3}{4}$$

(b) Since $N = 2$ we have $\Delta x = \frac{b-a}{N} = \frac{2-0}{2} = 1$. The interval $[0, 2]$ is partitioned into the intervals $[0, 1]$ and $[1, 2]$. Thus, the Trapezoidal estimate of the integral is

$$T_2 = \frac{1}{2}\Delta x\,[f(0) + 2f(1) + f(2)]$$
$$T_2 = \frac{1}{2}\left[\frac{1}{2(0)+1} + 2 \cdot \frac{1}{2(1)+1} + \frac{1}{2(2)+1}\right]$$
$$T_2 = \frac{1}{2}\left[1 + \frac{2}{3} + \frac{1}{5}\right]$$
$$T_2 = \frac{14}{15}$$

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Problem 5 Solution

5. Determine whether or not the following infinite series converge. Justify your answers.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k-1}{k^3+5}$

(b) $\displaystyle\sum_{k=3}^{\infty} \frac{1}{(\ln k)^{10}}$

Solution:

(a) Let $a_k = \frac{k-1}{k^3+5}$ and $b_k = \frac{1}{k^2}$. The series $\sum b_k$ is a convergent p-series. Moreover,

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{a_k}{b_k} \\
L &= \lim_{n\to\infty} \frac{\frac{k-1}{k^3+5}}{\frac{1}{k^2}} \\
L &= \lim_{n\to\infty} \frac{k^3-k^2}{k^3+5} \\
L &= \lim_{n\to\infty} \frac{k^3-k^2}{k^3+5} \cdot \frac{\frac{1}{k^3}}{\frac{1}{k^3}} \\
L &= \lim_{n\to\infty} \frac{1-\frac{1}{k}}{1+\frac{5}{k^3}} \\
L &= \frac{1-0}{1+0} \\
L &= 1
\end{aligned}$$

Since $0 < L < \infty$ and $\sum b_k$ converges, the series $\sum a_k$ converges by the Limit Comparison Test.

(b) Let $a_k = \frac{1}{(\ln k)^{10}}$ and $b_k = \frac{1}{k}$. The series $\sum b_k$ is a divergent p-series. Moreover,

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{a_k}{b_k} \\
L &= \lim_{n\to\infty} \frac{\frac{1}{(\ln k)^{10}}}{\frac{1}{k}} \\
L &= \lim_{n\to\infty} \frac{k}{(\ln k)^{10}} \\
L &= \infty
\end{aligned}$$

using the fact that $(\ln k)^{10} \ll k$ as $k \to \infty$. Since $L = \infty$ and $\sum b_k$ diverges, the series $\sum a_k$ diverges by the Limit Comparison Test.