Math 181, Exam 2, Fall 2014
Problem 1 Solution

1. **Integrals, Part I (Trigonometric integrals: 6 points).** Evaluate the integral:

$$\int \sin^3(x)\sqrt{\cos(x)}\,dx.$$

Solution: We begin by rewriting $\sin^3(x)$ as

$$\sin^3(x) = \sin(x)\sin^2(x) = \sin(x)(1-\cos^2(x)).$$

Then, after using the substitution

$$u = \cos(x), \quad -du = \sin(x)\,dx$$

we have the following result:

$$\begin{aligned}
\int \sin^3(x)\sqrt{\cos(x)}\,dx &= \int \sin(x)(1-\cos^2(x))\sqrt{\cos(x)}\,dx, \\
&= \int \underbrace{(1-\cos^2(x))}_{1-u^2}\underbrace{\sqrt{\cos(x)}}_{\sqrt{u}}\,\underbrace{\sin(x)\,dx}_{-du}, \\
&= \int (1-u^2)\sqrt{u}\,(-du), \\
&= -\int (1-u^2)u^{1/2}\,du, \\
&= -\int \left(u^{1/2}-u^{5/2}\right)du, \\
&= -\frac{2}{3}u^{3/2} + \frac{2}{7}u^{7/2} + C, \\
&= -\frac{2}{3}(\cos(x))^{3/2} + \frac{2}{7}(\cos(x))^{7/2} + C.
\end{aligned}$$

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Problem 2 Solution

2. **Integrals, Part II (Trigonometric substitutions: 6 points).** Evaluate the following integral. Do not forget to simplify your answer completely – the final answer should not contain inverse trigonometric functions!

$$\int \frac{1}{(4+x^2)^{3/2}}\,dx.$$

Solution: We begin with the substitition

$$x = 2\tan\theta, \quad dx = 2\sec^2\theta\,d\theta$$

Using the above substitution, the integral transforms as follows:

$$\begin{aligned}
\int \frac{1}{(4+x^2)^{3/2}}\,dx &= \int \frac{1}{(4+(2\tan\theta)^2)^{3/2}}\,2\sec^2\theta\,d\theta \\
&= \int \frac{1}{(\underbrace{4+4\tan^2\theta}_{4\sec^2\theta})^{3/2}}\,2\sec^2\theta\,d\theta \\
&= \int \frac{1}{(4\sec^2\theta)^{3/2}}\,2\sec^2\theta\,d\theta \\
&= \int \frac{1}{4^{3/2}(\sec^2\theta)^{3/2}}\,2\sec^2\theta\,d\theta \\
&= \int \frac{1}{8\sec^3\theta}\,2\sec^2\theta\,d\theta \\
&= \frac{1}{4}\int \frac{1}{\sec\theta}\,d\theta \\
&= \frac{1}{4}\int \cos\theta\,d\theta
\end{aligned}$$

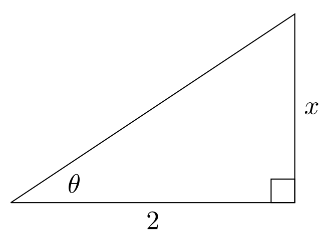
The result of the above integral is

$$\int \frac{1}{(4+x^2)^{3/2}}\,dx = \frac{1}{4}\int \cos\theta\,d\theta = \frac{1}{4}\sin\theta + C$$

Finally, we write the answer in terms of x. To do this, we return to the substitution $x = 2\tan\theta$ and rewrite the equation as

$$\tan\theta = \frac{x}{2} = \frac{\text{opposite}}{\text{adjacent}}$$

where "opposite" is the side opposite the angle θ in the right triangle below and "adjacent" is the side adjacent to θ.



The hypotenuse of the right triangle is $\sqrt{x^2+4}$ by way of Pythagoras' Theorem. Thus, an expression for $\sin\theta$ in terms of x is

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{\sqrt{x^2+4}}.$$

Our final answer is now

$$\int \frac{1}{(4+x^2)^{3/2}}\,dx = \frac{1}{4}\sin\theta + C = \frac{1}{4}\cdot\frac{x}{\sqrt{x^2+4}} + C.$$

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Problem 3 Solution

3. **Integrals, Part III (Partial fractions decompositions: 8 points).** Evaluate the integral:

$$\int \frac{1}{x^2+x-6}\,dx.$$

Solution: The partial fraction decomposition method will be used to evaluate the integral. The denominator factors into:

$$x^2+x-6=(x+3)(x-2)$$

Since the denominator is a product of distinct linear factors, the integrand may be decomposed as follows:

$$\frac{1}{x^2+x-6}=\frac{A}{x+3}+\frac{B}{x-2}$$

We find the unknown constants A and B by first clearing denominators:

$$1=A(x-2)+B(x+3)$$

and then making two substitutions:

- $x=2 \Longrightarrow 1=A(2-2)+B(2+3)$ which yields $B=\frac{1}{5}$
- $x=-3 \Longrightarrow 1=A(-3-2)+B(-3+3)$ which yields $A=-\frac{1}{5}$

Substituting these values into the decomposition and integrating yields the following result:

$$\int \frac{1}{x^2+x-6}\,dx=\int\left(\frac{-\frac{1}{5}}{x+3}+\frac{\frac{1}{5}}{x-2}\right)dx=-\frac{1}{5}\ln|x+3|+\frac{1}{5}\ln|x-2|+C$$

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Problem 4 Solution

4. **Integrals, Part IV (Improper integrals: 6 points).** Evaluate the following improper integral or show that it diverges.

$$\int_0^\infty xe^{-x}\,dx.$$

Solution: The upper limit of integration makes the integral improper. Thus, we replace the upper limit with b and take the limit of the integral as $b \to \infty$.

$$\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx. \qquad \text{[1 point]}$$

The integral may be evaluated using integration by parts. We make the following definitions:

$$u = x, \quad dv = e^{-x}\,dx$$

which yield

$$du = dx, \quad v = -e^{-x}.$$

Thus, using the integration by parts formula, we have

$$\begin{aligned}
\int_0^b u\,dv &= uv\Big|_0^b - \int_0^b v\,du \\
\int_0^b xe^{-x}\,dx &= x(-e^{-x})\Big|_0^b - \int_0^b (-e^{-x})\,dx \\
&= -xe^{-x}\Big|_0^b + \int_0^b e^{-x}\,dx \\
&= -xe^{-x} - e^{-x}\Big|_0^b \\
&= -e^{-x}(x+1)\Big|_0^b \\
&= -\frac{x+1}{e^x}\Big|_0^b \\
&= -\frac{b+1}{e^b} + 1
\end{aligned}$$

The value of the improper integral is then

$$\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx = \lim_{b\to\infty}\left(\underbrace{-\frac{b+1}{e^b}}_{\to 0} + 1\right) = 1$$

where the limit of $\dfrac{b+1}{e^b}$ as $b \to \infty$ is 0 since $b+1 \ll e^b$ for large b.

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Problem 5 Solution

5. **Integrals, Part V (Numerical integration: 8 points).** Evaluate the integral

$$\int_0^2 \frac{1}{(2x+1)^2}\,dx$$

using:

(a) analytical methods, to obtain the exact solution;

(b) the Midpoint Method for numeral integration, with $n = 2$ subintervals. Simplify your answer!

Compare the two results by computing the absolute and the relative errors.

Solution:

(a) The Fundamental Theorem of Calculus yields the following exact solution:

$$\int_0^2 \frac{1}{(2x+1)^2}\,dx = \left[-\frac{1}{2}\cdot\frac{1}{2x+1}\right]_0^2 = -\frac{1}{10}+\frac{1}{2} = \frac{2}{5}.$$

(b) The width of each subinterval of $[0,2]$ is given by

$$\Delta x = \frac{2-0}{2} = 1$$

The two subintervals of $[0,2]$ are $[0,1]$ and $[1,2]$. The corresponding midpoints of these subintervals are

$$m_1 = \frac{1}{2}, \quad m_2 = \frac{3}{2}$$

The value of M_2 is then

$$\begin{aligned} M_2 &= \Delta x\,[f(m_1)+f(m_2)] \\ &= 1\cdot\left[f\left(\frac{1}{2}\right)+f\left(\frac{3}{2}\right)\right] \\ &= \frac{1}{(2\cdot\frac{1}{2}+1)^2}+\frac{1}{(2\cdot\frac{3}{2}+1)^2} \\ &= \frac{1}{4}+\frac{1}{16} \\ &= \frac{5}{16} \end{aligned}$$

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Problem 6 Solution

6. **Sequences (5 points).** For each of the following sequences, determine whether they have a limit or not, and if yes, then compute the limit.

(a) $a_n = \frac{2n+1}{n^2-2}$ for all $n \geq 1$;

(b) $b_n = \sin(n\pi)$ for all $n \geq 1$.

Solution:

(a) The limit of the sequence is

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n+1}{n^2-2} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{2}{n}+\frac{1}{n^2}}{1-\frac{2}{n^2}} = \frac{0+0}{1-0} = 0$$

(b) The terms of the sequence are

$$\{a_n\} = \{\sin(\pi), \sin(2\pi), \sin(3\pi), \ldots, \sin(n\pi), \ldots\} = \{0, 0, 0, \ldots, 0, \ldots\}$$

Thus, the limit of the sequence is 0.

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Problem 7 Solution

7. Integrals, Part VI (7 points).

(a) Decompose the polynomial $q(x) = x^3 - x^2 + 2x - 2$ into a product of irreducible factors, noticing that $q(1) = 0$.

(b) Decompose the rational function $r(x) = \frac{x+2}{x^3-x^2+2x-2}$ into its partial fraction decomposition.

(c) Evaluate $\int r(x)\,dx$.

Solution:

(a) We can factor $q(x)$ as follows:

$$q(x) = (x^3 - x^2) + (2x - 2) = x^2(x-1) + 2(x-1) = (x^2+2)(x-1)$$

where $x^2 + 2$ is irreducible because its discriminant, $b^2 - 4ac = 0^2 - 4(1)(2) = -8$, is negative.

(b) The partial fraction decomposition of $r(x)$ is

$$\frac{x+2}{x^3 - x^2 + 2x - 2} = \frac{A}{x-1} + \frac{Bx + C}{x^2 + 2}$$

Clearing denominators gives us

$$x + 2 = A(x^2 + 2) + (Bx + C)(x - 1)$$

Expanding the right hand side and collecting like terms results in the equation

$$\begin{aligned} x + 2 &= (A + B)x^2 + (C - B)x + (2A - C) \\ 0x^2 + 1x + 2 &= (A + B)x^2 + (C - B)x + (2A - C) \end{aligned}$$

Equating coefficients of x^n on both sides of the above equation gives us the following system of equations

$$\begin{aligned} x^2 &: \quad A + B = 0, \\ x^1 &: \quad C - B = 1, \\ x^0 &: \quad 2A - C = 2. \end{aligned}$$

The solution to the system is $C = 0$, $B = -1$, and $A = 1$.

(c) The integral of $r(x)$ is then

$$\int r(x)\,dx = \int \left(\frac{1}{x-1} - \frac{1}{x^2+2} \right) dx = \int \frac{1}{x-1}\,dx - \int \frac{x}{x^2+2}\,dx$$

In the second integral on the right hand side above, we let $u = x^2 + 2$. Then $du = 2x\,dx \quad \Rightarrow \quad \frac{1}{2}\,du = x\,dx$ and we have

$$\int \frac{x}{x^2+2}\,dx = \frac{1}{2} \int \frac{1}{u}\,du = \frac{1}{2} \ln|u| = \frac{1}{2}\ln(x^2+2)$$

Thus, we have

$$\int r(x)\,dx = \int \frac{1}{x-1}\,dx - \int \frac{1}{x^2+2}\,dx = \ln|x-1| - \frac{1}{2}\ln(x^2+2) + C$$

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Problem 8 Solution

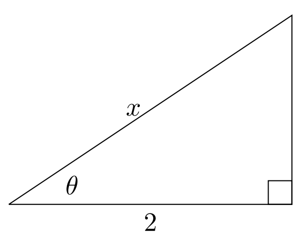
8. **Integrals, Part VII (6 points)**. Evaluate the following integral:

$$\int \frac{1}{x^2\sqrt{x^2-4}}\,dx.$$

Solution: Let $x = 2\sec\theta$ and $dx = 2\sec\theta\tan\theta\,d\theta$. Then

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2-4}}\,dx &= \int \frac{1}{(2\sec\theta)^2\sqrt{(2\sec\theta)^2-4}}\,(2\sec\theta\tan\theta\,d\theta) \\
&= \int \frac{2\sec\theta\tan\theta}{4\sec^2\theta\sqrt{4\sec^2\theta-4}}\,d\theta \\
&= \int \frac{\tan\theta}{2\sec\theta\sqrt{4(\sec^2\theta-1)}}\,d\theta \\
&= \int \frac{\tan\theta}{2\sec\theta\sqrt{4}\sqrt{\sec^2\theta-1}}\,d\theta \\
&= \int \frac{\tan\theta}{2\sec\theta\cdot 2\cdot\sqrt{\tan^2\theta}}\,d\theta \\
&= \int \frac{\tan\theta}{4\sec\theta\tan\theta}\,d\theta \\
&= \frac{1}{4}\int \cos\theta\,d\theta \\
&= \frac{1}{4}\sin\theta + C
\end{aligned}$$

Since $x = 2\sec\theta$ we know that $\sec\theta = \frac{x}{2}$ which means that $\cos\theta = \frac{2}{x}$. We set up the following right triangle



Using Pythagoras' Theorem, the side opposite θ is $\sqrt{x^2-4}$. Therefore,

$$\int \frac{1}{x^2\sqrt{x^2-4}}\,dx = \frac{1}{4}\sin\theta + C = \frac{1}{4}\cdot\frac{\sqrt{x^2-4}}{x} + C$$