Math 181, Exam 2, Spring 2006
Problem 1 Solution

1. Compute the indefinite integral:

$$\int \frac{dx}{x^2 + 4x + 5}$$

Solution: We begin by completing the square.

$$\int \frac{dx}{x^2 + 4x + 5} = \int \frac{dx}{(x+2)^2 + 1}$$

We now evaluate the integral using the u-substitution method. Let $u = x+2$. Then $du = dx$ and we get:

$$\begin{aligned} \int \frac{dx}{x^2 + 4x + 5} &= \int \frac{dx}{(x+2)^2 + 1} \\ &= \int \frac{du}{u^2 + 1} \\ &= \arctan u + C \\ &= \boxed{\arctan(x+2) + C} \end{aligned}$$

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Problem 2 Solution

2. Determine if the following improper integrals converge or not. If they do compute their value.

$$\int_0^{+\infty} xe^{-x}\,dx \qquad \int_1^{+\infty} \frac{x+1}{x^2+x+1}\,dx$$

Solution: Each integral is improper due to the infinite upper limit of integration. We evaluate the first integral by turning it into a limit calculation.

$$\int_0^{+\infty} xe^{-x}\,dx = \lim_{R\to+\infty} \int_0^R xe^{-x}\,dx$$

We use Integration by Parts to compute the integral. Let $u = x$ and $v' = e^{-x}$. Then $u' = 1$ and $v = -e^{-x}$. Using the Integration by Parts formula we get:

$$\begin{aligned}
\int_a^b uv'\,dx &= \Big[uv\Big]_a^b - \int_a^b u'v\,dx \\
\int_0^R xe^{-x}\,dx &= \Big[-xe^{-x}\Big]_0^R - \int_0^R \left(-e^{-x}\right)\,dx \\
&= \Big[-xe^{-x}\Big]_0^R + \int_0^R e^{-x}\,dx \\
&= \Big[-xe^{-x}\Big]_0^R + \Big[-e^{-x}\Big]_0^R \\
&= \Big[-Re^{-R} + 0e^{-0}\Big] + \Big[-e^{-R} + e^{-0}\Big] \\
&= -\frac{R}{e^R} - \frac{1}{e^R} + 1
\end{aligned}$$

We now take the limit of the above function as $R \to +\infty$.

$$\begin{aligned}
\int_0^{+\infty} xe^{-x}\,dx &= \lim_{R\to+\infty} \int_0^R xe^{-x}\,dx \\
&= \lim_{R\to+\infty} \left(-\frac{R}{e^R} - \frac{1}{e^R} + 1\right) \\
&= -\lim_{R\to+\infty} \frac{R}{e^R} - \lim_{R\to+\infty} \frac{1}{e^R} + 1 \\
&= -\lim_{R\to+\infty} \frac{R}{e^R} - 0 + 1 \\
&\overset{\text{L'H}}{=} -\lim_{R\to+\infty} \frac{(R)'}{(e^R)'} - 0 + 1 \\
&= -\lim_{R\to+\infty} \frac{1}{e^R} - 0 + 1 \\
&= -0 - 0 + 1 \\
&= \boxed{1}
\end{aligned}$$

We will show that the second integral diverges using the Comparison Test. Let $f(x) = \frac{x+1}{x^2+x+1}$. We must find a function $g(x)$ such that:

$$(1)\ \int_1^{+\infty} g(x)\,dx \text{ diverges} \quad \text{and} \quad (2)\ 0 \le g(x) \le f(x) \text{ for } x \ge 1$$

We choose the function $g(x)$ by using the fact that $0 \le x \le x+1$ and $0 \le x^2+x+1 \le x^2+x^2+x^2$ for $x \ge 1$. Then we get:

$$0 \le \frac{x}{x^2+x^2+x^2} \le \frac{x+1}{x^2+x+1}$$
$$0 \le \frac{x}{3x^2} \le \frac{x+1}{x^2+x+1}$$
$$0 \le \frac{1}{3x} \le \frac{x+1}{x^2+x+1}$$

So we choose $g(x) = \frac{1}{3x}$ so that $0 \le g(x) \le f(x)$ for $x \ge 1$. Furthermore, we know that:

$$\int_1^{+\infty} g(x)\,dx = \int_1^{+\infty} \frac{1}{3x}\,dx = \frac{1}{3}\int_1^{+\infty} \frac{1}{x}\,dx$$

diverges because this is a p-integral with $p = 1 \le 1$. Thus, the integral $\int_1^{+\infty} \frac{x+1}{x^2+x+1}\,dx$ **diverges** by the Comparison Test.

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Problem 3 Solution

3. Let R be the region in the xy-plane bounded by $y = 0$ and $y = x - x^2$ for $0 \leq x \leq 1$.

   i) Set up an integral to evaluate the volume of the solid obtained by revolving the region R about the line $y = -3$.

   ii) Compute the integral.

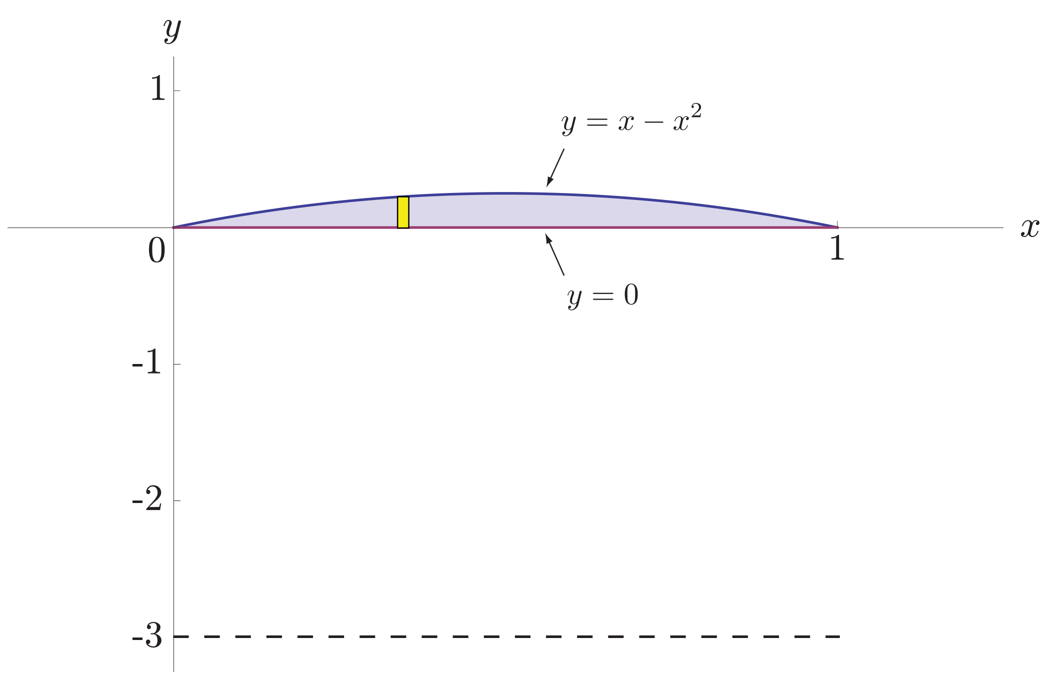
Solution:

i) We find the volume of the solid using the **Washer Method**. The variable of integration is x and the corresponding formula is:

$$V = \pi \int_a^b \left[(\text{top} - (-3))^2 - (\text{bottom} - (-3))^2 \right] dx$$

The top curve is $y = x - x^2$ and the bottom curve is $y = 0$. The lower limit of integration is $x = 0$ and the upper limit is $x = 1$. The integral that represents the volume is:

$$V = \pi \int_0^1 \left[\left(x - x^2 - (-3)\right)^2 - (0 - (-3))^2 \right] dx$$



ii) The volume is:

$$\begin{aligned}
V &= \pi \int_0^1 \left[\left(x - x^2 - (-3)\right)^2 - (0 - (-3))^2 \right] \, dx \\
&= \pi \int_0^1 \left[\left(x - x^2 + 3\right)^2 - (0 + 3)^2 \right] \, dx \\
&= \pi \int_0^1 \left(6x - 5x^2 - 2x^3 + x^4\right) \, dx \\
&= \pi \left[3x^2 - \frac{5}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5 \right]_0^1 \\
&= \pi \left(3 - \frac{5}{3} - \frac{1}{2} + \frac{1}{5} \right) \\
&= \boxed{\frac{31\pi}{30}}
\end{aligned}$$

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Problem 4 Solution

4. A retail store chain conducted a customer satisfaction survey. Each completed questionnaire was processed and produced a satisfaction level t between 0 (complete disappointment) and 1 (complete satisfaction). The subsequent analysis showed that the density function of the satisfaction level is given by $p(t) = 3t^2$ for $0 \leq t \leq 1$ (and 0 otherwise).

i) Find what percentage of customers registered satisfaction level between $\frac{1}{3}$ and $\frac{2}{3}$.

ii) Find the mean value of t.

iii) Find the median of t.

Solution:

i) The percentage of customers registering a satisfaction level between a and b is given by the formula:

$$\int_a^b p(t)\,dt$$

Using $a = \frac{1}{3}$, $b = \frac{2}{3}$, and $p(t) = 3t^2$ we have:

$$\int_a^b p(t)\,dt = \int_{1/3}^{2/3} 3t^2\,dt = \left[t^3\right]_{1/3}^{2/3} = \left(\frac{2}{3}\right)^3 - \left(\frac{1}{3}\right)^3 = \boxed{\frac{7}{27}}$$

ii) The mean value of t is given by the formula:

$$\int_a^b tp(t)\,dt$$

Using $a = 0$, $b = 1$, and $p(t) = 3t^2$ we have:

$$\int_a^b tp(t)\,dt = \int_0^1 t\left(3t^2\right)\,dt = \int_0^1 3t^3\,dt = \left[\frac{3}{4}t^4\right]_0^1 = \boxed{\frac{3}{4}}$$

iii) The median of t is the value of T such that

$$\int_a^T p(t)\,dt = \frac{1}{2}$$

Using $a = 0$ and $p(t) = 3t^2$ we have:

$$\begin{aligned}
\int_a^T p(t)\,dt &= \frac{1}{2} \\
\int_0^T 3t^2\,dt &= \frac{1}{2} \\
\left[t^3\right]_0^T &= \frac{1}{2} \\
T^3 &= \frac{1}{2} \\
T &= \boxed{\frac{1}{\sqrt[3]{2}}}
\end{aligned}$$

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Problem 5 Solution

5. Compute the sum of the following series:

$$\sum_{n=2}^{+\infty} \frac{2}{3^{n+3}}$$

Solution: We recognize the given series as a geometric series. In order to find its sum we must first rewrite the series.

$$\sum_{n=2}^{+\infty} \frac{2}{3^{n+3}} = \sum_{n=2}^{+\infty} \frac{2}{3^n 3^3} = \sum_{n=2}^{+\infty} \frac{2}{3^3} \cdot \frac{1}{3^n} = \sum_{n=2}^{+\infty} \frac{2}{27} \left(\frac{1}{3}\right)^n$$

This is a convergent geometric series because $|r| = |\frac{1}{3}| < 1$. We can now use the formula:

$$\sum_{n=M}^{+\infty} cr^n = r^M \cdot \frac{c}{1-r}$$

where $M = 2$, $c = \frac{2}{27}$, and $r = \frac{1}{3}$. The sum of the series is then:

$$\sum_{n=2}^{+\infty} \frac{2}{27} \left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^2 \cdot \frac{\frac{2}{27}}{1 - \frac{1}{3}} = \boxed{\frac{1}{81}}$$

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Problem 6 Solution

6. Determine whether the following series converge or not:

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n+1} n^3}{2^n}, \qquad \sum_{n=1}^{+\infty} \frac{3+2^{-n}}{\sqrt{n}}, \qquad \sum_{n=1}^{+\infty} \frac{(-1)^{-n}}{\sqrt{n^2+n+1}}$$

Solution: The first series is alternating. We check for absolute convergence by considering the series of absolute values:

$$\sum_{n=1}^{+\infty} \left| \frac{(-1)^{n+1} n^3}{2^n} \right| = \sum_{n=1}^{+\infty} \frac{n^3}{2^n}$$

We use the Ratio Test to determine whether or not this series converges.

$$\begin{aligned} \rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\ &= \lim_{n\to\infty} \frac{(n+1)^3}{2^{n+1}} \cdot \frac{2^n}{n^3} \\ &= \lim_{n\to\infty} \frac{1}{2} \left(\frac{n+1}{n} \right)^3 \\ &= \lim_{n\to\infty} \frac{1}{2} \left(1 + \frac{1}{n} \right)^3 \\ &= \frac{1}{2} \end{aligned}$$

Since $\rho = \frac{1}{2} < 1$, the series of absolute values converges by the Ratio Test. Therefore, the series $\sum_{n=1}^{+\infty} \frac{(-1)^{n+1} n^3}{2^n}$ **converges**.

Note: Here are the results of using the other convergence tests to determine whether or not $\sum_{n=1}^{+\infty} \frac{n^3}{2^n}$ converges.

(1) The Divergence Test fails because $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^3}{2^n} = 0$.

(2) The Integral Test will show that the series converges because the improper integral

$$\int_1^{\infty} x^3 2^{-x}\, dx = \frac{6 + (\ln 2)^2 (3 + \ln 2) + \ln 64}{2(\ln 2)^4}$$

converges. However, in order to get the above result, we must integrate by parts three times.

(3) The Comparison Test with $\sum_{n=1}^{+\infty} r^n$, using any value of r satisfying $\frac{1}{2} < r < 1$, shows that the series converges.

(4) The Limit Comparison Test with $\sum_{n=1}^{+\infty} r^n$, using any value of r satisfying $\frac{1}{2} < r < 1$, or with $\sum_{n=1}^{+\infty} \frac{1}{n^p}$, using any $p > 1$, shows that the series converges.

(5) The Root Test shows that the series converges because $\rho = \lim_{n\to\infty} \sqrt[n]{\frac{n^3}{2^n}} = \frac{1}{2} < 1$. However, the limit calculation is not straightforward.

Rewriting the second series we have:

$$\sum_{n=1}^{+\infty} \frac{3 + 2^{-n}}{\sqrt{n}} = 3\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n}} + \sum_{n=1}^{+\infty} \frac{1}{2^n\sqrt{n}}$$

The series $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n}}$ diverges because it is a p-series with $p = \frac{1}{2} < 1$. Therefore, the series $\sum_{n=1}^{+\infty} \frac{3+2^{-n}}{\sqrt{n}}$ **diverges**.

The third series is alternating so we test for convergence using the Leibniz Test. Let $a_n = f(n) = \frac{1}{\sqrt{n^2+n+1}}$. The function $f(n)$ is decreasing for $n \geq 1$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n^2 + n + 1}} = 0$$

Therefore, the series $\sum_{n=1}^{+\infty} \frac{(-1)^{-n}}{\sqrt{n^2+n+1}}$ **converges** by the Leibniz Test.

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Problem 7 Solution

7. Determine the radius of convergence of the power series:

$$\sum_{n=1}^{+\infty} \frac{(-2)^n(x-2)^n}{n^4}$$

Solution: We determine the radius of convergence using the Ratio Test.

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \left|\frac{(-2)^{n+1}(x-2)^{n+1}}{(n+1)^4} \cdot \frac{n^4}{(-2)^n(x-2)^n}\right| \\
&= \lim_{n\to\infty} \left|\frac{(-2)^{n+1}}{(-2)^n} \cdot \frac{n^4}{(n+1)^4} \cdot \frac{(x-2)^{n+1}}{(x-2)^n}\right| \\
&= \lim_{n\to\infty} \left|(-2)^1 \left(\frac{n}{n+1}\right)^4 (x-2)\right| \\
&= \lim_{n\to\infty} \left|2 \left(\frac{1}{1+\frac{1}{n}}\right)^4 (x-2)\right| \\
&= 2|x-2| \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^4 \\
&= 2|x-2|
\end{aligned}$$

In order to achieve convergence, it must be the case that $\rho = 2|x-2| < 1$. Therefore, $|x-2| < \frac{1}{2}$ and the radius of convergence is $\boxed{\frac{1}{2}}$.