Math 181, Exam 2, Spring 2008
Problem 1 Solution

1. Compute the indefinite integral:

$$\int \frac{8}{\sqrt{4-x^2}}\,dx$$

Solution: We evaluate the integral using the u-substitution method. Let $u = \frac{x}{2}$. Then $du = \frac{1}{2}\,dx \quad \Rightarrow \quad 2\,du = dx$ and $x = 2u$ and we get:

$$\begin{aligned}
\int \frac{8}{\sqrt{4-x^2}}\,dx &= \int \frac{8}{\sqrt{4-(2u)^2}}\,(2\,du) \\
&= \int \frac{16}{\sqrt{4-4u^2}}\,du \\
&= \int \frac{16}{\sqrt{4}\sqrt{1-u^2}}\,du \\
&= 8\int \frac{1}{\sqrt{1-u^2}}\,du \\
&= 8\arcsin u + C \\
&= \boxed{8\arcsin\left(\frac{x}{2}\right) + C}
\end{aligned}$$

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Problem 2 Solution

2. Consider the curve $y = \frac{1}{2}x^2$ over the interval $0 \le x \le 1$.

(a) Write the integral you would compute to find the length of the curve (**do not solve the integral**).

(b) Use the Trapezoidal Rule with $n = 1$ to estimate the value of the integral from part (a).

Solution:

(a) The arclength formula is:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

where $\frac{dy}{dx} = (\frac{1}{2}x^2)' = x$, $a = 0$, and $b = 1$. Therefore, the integral we would compute to find the length of the curve is

$$\boxed{L = \int_0^1 \sqrt{1 + x^2}\, dx}$$

(b) We now use the Trapezoidal Rule with $n = 1$ to estimate the value of the integral. The formula we will use is:

$$T_1 = \frac{\Delta x}{2}\left[f(0) + f(1)\right]$$

where $f(x) = \sqrt{1 + x^2}$ and the value of Δx is:

$$\Delta x = \frac{b - a}{n} = \frac{1 - 0}{1} = 1$$

The value of T_1 is then:

$$\begin{aligned} T_1 &= \frac{\Delta x}{2}\left[f(0) + f(1)\right] \\ &= \frac{1}{2}\left[\sqrt{1 + 0^2} + \sqrt{1 + 1^2}\right] \\ &= \boxed{\frac{1}{2}\left(1 + \sqrt{2}\right)} \end{aligned}$$

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Problem 3 Solution

3. Show whether the integral below diverges or converges. If it converges, find the value of the integral:

$$\int_3^{\infty} \frac{dx}{\sqrt{x+1}}$$

Solution: We evaluate the integral by turning it into a limit calculation.

$$\int_3^{\infty} \frac{dx}{\sqrt{x+1}} = \lim_{R\to\infty} \int_3^{R} \frac{dx}{\sqrt{x+1}}$$

We use the u-substitution method to compute the integral. Let $u = x+1$ and $du = dx$. The indefinite integral then becomes

$$\int \frac{dx}{\sqrt{x+1}} = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} = 2\sqrt{x+1}$$

The definite integral from 3 to R is

$$\begin{aligned}\int_3^{R} \frac{dx}{\sqrt{x+1}} &= \left[2\sqrt{x+1}\right]_3^R \\ &= 2\sqrt{R+1} - 2\sqrt{3+1} \\ &= 2\sqrt{R+1} - 4\end{aligned}$$

Taking the limit as $R \to \infty$ we get

$$\begin{aligned}\int_3^{\infty} \frac{dx}{\sqrt{x+1}} &= \lim_{R\to\infty} \int_3^{R} \frac{dx}{\sqrt{x+1}} \\ &= \lim_{R\to\infty} \left(2\sqrt{R+1} - 4\right) \\ &= \infty - 4 \\ &= \infty\end{aligned}$$

Therefore, the integral **diverges**.

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Problem 4 Solution

4. Compute the indefinite integral:

$$\int x^2 \ln x \, dx$$

Solution: We compute the integral using Integration by Parts. Let $u = \ln x$ and $v' = x^2$. Then $u' = \dfrac{1}{x}$ and $v = \dfrac{1}{3}x^3$. Using the Integration by Parts formula:

$$\int uv' \, dx = uv - \int u'v \, dx$$

we get:

$$\begin{aligned} \int x^2 \ln x \, dx &= (\ln x)\left(\frac{1}{3}x^3\right) - \int \frac{1}{x} \cdot \frac{1}{3}x^3) \, dx \\ &= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 \, dx \\ &= \boxed{\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C} \end{aligned}$$

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Problem 5 Solution

5. Complete each of the following:

(a) Find the Taylor polynomial of degree 2 for the function $f(x) = \ln x$ about $x = 1$.

(b) Calculate an upper bound on the error $|f(1.1) - T_2(1.1)|$ using the Error Bound formula. **You should not simplify your answer.**

Solution:

(a) The degree two Taylor polynomial for $f(x) = \ln x$ around $x = 1$ has the formula:

$$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$$

The derivatives of $f(x)$ evaluated at $x = 1$ are:

k	$f^{(k)}(x)$	$f^{(k)}(1)$
0	$\ln x$	$\ln 1 = 0$
1	$\frac{1}{x}$	$\frac{1}{1} = 1$
2	$-\frac{1}{x^2}$	$-\frac{1}{1^2} = -1$

The Taylor polynomial $T_2(x)$ is then:

$$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$$

$$T_2(x) = 0 + (x-1) - \frac{1}{2!}(x-1)^2$$

$$\boxed{T_2(x) = (x-1) - \frac{1}{2}(x-1)^2}$$

(b) The error bound for part (a) is given by the formula:

$$\text{Error} \leq K\frac{|x-a|^{n+1}}{(n+1)!}$$

where $x = 1.1$, $a = 1$, $n = 2$, and K satisfies the inequality $|f'''(u)| \leq K$ for all $u \in [1, 1.1]$. Since $f'''(x) = \frac{2}{x^3}$, we conclude that $|f'''(u)| = |\frac{2}{u^3}| < 2$ for all $u \in [1, 1.1]$. So we choose $K = 2$ and the error bound is:

$$\text{Error} \leq 2\frac{|1.1-1|^3}{3!} = \boxed{\frac{1}{3000}}$$

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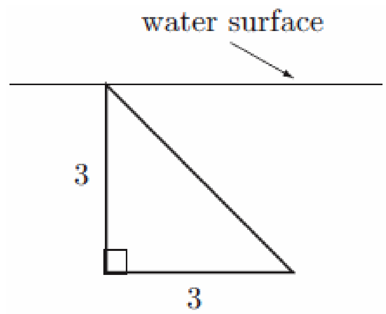
Problem 6 Solution

6. A right isosceles triangular plate is vertically submerged below the surface of a fluid of weight density w. The top of the plate is at the surface of the fluid. Find the fluid force on the plate in terms of w.



Solution: We put the origin of the coordinate system at the vertex of the triangle at the water surface and define the positive y direction as being downward. The fluid force is then:

$$F = w \int_a^b y f(y)\, dy$$

where $a = 0$, $b = 3$, and $f(y) = y$ is the length of a horizontal strip of the plate at a depth of y from the water surface. The fluid force is then:

$$F = w \int_0^3 y(y)\, dy$$
$$F = w \int_0^3 y^2\, dy$$
$$F = w \left[\frac{1}{3} y^3 \right]_0^3$$
$$\boxed{F = 9w}$$

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Problem 7 Solution

7. Compute the indefinite integral:

$$\int \frac{2}{x(x^2+1)}\,dx$$

Solution: We will evaluate the integral using Partial Fraction Decomposition. First, we decompose the rational function into a sum of simpler rational functions.

$$\frac{2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Next, we multiply the above equation by $x(x^2+1)$ to get:

$$2 = A(x^2+1) + (Bx+C)x$$

Then we plug in three different values for x to create a system of three equations in three unknowns (A, B, C). We select $x = 0$, $x = 1$, and $x = -1$ for simplicity.

$$\begin{aligned}
x = 0: &\quad A(0^2+1) + (B(0)+C)(0) = 2 \quad \Rightarrow \quad A = 2 \\
x = 1: &\quad (2)(1^2+1) + (B(1)+C)(1) = 2 \quad \Rightarrow \quad B + C = -2 \\
x = -1: &\quad (2)((-1)^2+1) + (B(-1)+C)(-1) = 2 \quad \Rightarrow \quad B - C = -2
\end{aligned}$$

The solution to this system is $A = 2$, $B = -2$ and $C = 0$. Finally, we plug these values for A, B, and C back into the decomposition and integrate.

$$\begin{aligned}
\int \frac{2}{x(x^2+1)}\,dx &= \int \left(\frac{2}{x} + \frac{-2x+0}{x^2+1}\right)dx \\
&= 2\int \frac{1}{x}\,dx - \int \frac{2x}{x^2+1}\,dx
\end{aligned}$$

To evaluate the second integral we use the u-substitution method. Let $u = x^2+1$. Then $du = 2x\,dx$ and we get:

$$\begin{aligned}
\int \frac{2}{x(x^2+1)}\,dx &= 2\int \frac{1}{x}\,dx - \int \frac{2x}{x^2+1}\,dx \\
&= 2\int \frac{1}{x}\,dx - \int \frac{1}{u}\,du \\
&= 2\ln|x| - \ln|u| + C \\
&= \boxed{2\ln|x| - \ln(x^2+1) + C}
\end{aligned}$$