Math 181, Exam 2, Spring 2010
Problem 1a Solution

1a. Compute the improper integral: $\displaystyle\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^3}}$.

Solution: The integral is improper because the integrand is undefined at $x = 1$. We evaluate the integral by turning it into a limit calculation.

$$\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^3}} = \lim_{R\to 1^-} \int_0^R \frac{x^2\,dx}{\sqrt{1-x^3}}$$

To compute the integral we use the u-substitution method with $u = 1 - x^3$. Then $-\frac{1}{3}\,du = x^2\,dx$ and we get:

$$\int \frac{x^2\,dx}{\sqrt{1-x^3}} = -\frac{1}{3}\int \frac{du}{\sqrt{u}} = -\frac{2}{3}\sqrt{u} = -\frac{2}{3}\sqrt{1-x^3}$$

The definite integral from 0 to R is:

$$\begin{aligned}\int_0^R \frac{x^2\,dx}{\sqrt{1-x^3}} &= \left[-\frac{2}{3}\sqrt{1-x^3}\right]_0^R \\ &= -\frac{2}{3}\sqrt{1-R^3} + \frac{2}{3}\sqrt{1-0^3} \\ &= -\frac{2}{3}\sqrt{1-R^3} + \frac{2}{3}\end{aligned}$$

Taking the limit as $R \to 1^-$ we get:

$$\begin{aligned}\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^3}} &= \lim_{R\to 1^-} \int_0^R \frac{x^2\,dx}{\sqrt{1-x^3}} \\ &= \lim_{R\to 1^-} \left(-\frac{2}{3}\sqrt{1-R^3} + \frac{2}{3}\right) \\ &= -\frac{2}{3}\sqrt{1-0^3} + \frac{2}{3} \\ &= \boxed{\frac{2}{3}}\end{aligned}$$

Math 181, Exam 2, Spring 2010
Problem 1b Solution

1b. Compute the improper integral: $\displaystyle\int_0^{+\infty} \frac{x\,dx}{x^4+1}$.

Solution: The integral is improper because the upper limit of integration is infinite. We evaluate the integral by turning it into a limit calculation.

$$\int_0^{+\infty} \frac{x\,dx}{x^4+1} = \lim_{R\to+\infty} \int_0^R \frac{x\,dx}{x^4+1}$$

To compute the integral we use the u-substitution method with $u = x^2$. Then $\frac{1}{2}\,du = x\,dx$ and we get:

$$\int \frac{x\,dx}{x^4+1} = \frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\arctan u = \frac{1}{2}\arctan\left(x^2\right)$$

The definite integral from 0 to R is:

$$\begin{aligned}\int_0^R \frac{x\,dx}{x^4+1} &= \left[\frac{1}{2}\arctan\left(x^2\right)\right]_0^R \\ &= \frac{1}{2}\arctan\left(R^2\right) - \frac{1}{2}\arctan\left(0^2\right) \\ &= \frac{1}{2}\arctan\left(R^2\right)\end{aligned}$$

Taking the limit as $R \to +\infty$ we get:

$$\begin{aligned}\int_0^{+\infty} \frac{x\,dx}{x^4+1} &= \lim_{R\to+\infty} \int_0^R \frac{x\,dx}{x^4+1} \\ &= \lim_{R\to+\infty} \left[\frac{1}{2}\arctan\left(x^2\right)\right] \\ &= \frac{1}{2}\cdot\frac{\pi}{2} \\ &= \boxed{\frac{\pi}{4}}\end{aligned}$$

Math 181, Exam 2, Spring 2010
Problem 2a Solution

2a. Compute the arclength of the graph of $y = 2x^{3/2}$ from $x = 0$ to $x = 1$.

Solution: The arclength is:

$$\begin{aligned} L &= \int_a^b \sqrt{1 + f'(x)^2}\, dx \\ &= \int_0^1 \sqrt{1 + \left(3x^{1/2}\right)^2}\, dx \\ &= \int_0^1 \sqrt{1 + 9x}\, dx \end{aligned}$$

We now use the u-substitution $u = 1 + 9x$. Then $\frac{1}{9}\, du = dx$, the lower limit of integration changes from 0 to 1, and the upper limit of integration changes from 1 to 10.

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + 9x}\, dx \\ &= \frac{1}{9} \int_1^{10} \sqrt{u}\, du \\ &= \frac{1}{9} \left[\frac{2}{3} u^{3/2}\right]_1^{10} \\ &= \frac{1}{9} \left[\frac{2}{3}(10)^{3/2} - \frac{2}{3}(1)^{3/2}\right] \\ &= \boxed{\frac{2}{27}\left[10^{3/2} - 1\right]} \end{aligned}$$

Math 181, Exam 2, Spring 2010
Problem 2b Solution

2b. Compute the arclength of the graph of $y = 2x^{3/2}$ from $x = 2$ to $x = 3$.

Solution: The arclength is:

$$\begin{aligned} L &= \int_a^b \sqrt{1 + f'(x)^2}\, dx \\ &= \int_2^3 \sqrt{1 + \left(3x^{1/2}\right)^2}\, dx \\ &= \int_2^3 \sqrt{1 + 9x}\, dx \end{aligned}$$

We now use the u-substitution $u = 1 + 9x$. Then $\frac{1}{9}\, du = dx$, the lower limit of integration changes from 2 to 19, and the upper limit of integration changes from 3 to 28.

$$\begin{aligned} L &= \int_{19}^{28} \sqrt{1 + 9x}\, dx \\ &= \frac{1}{9} \int_{19}^{28} \sqrt{u}\, du \\ &= \frac{1}{9} \left[\frac{2}{3} u^{3/2} \right]_{19}^{28} \\ &= \frac{1}{9} \left[\frac{2}{3} (28)^{3/2} - \frac{2}{3} (19)^{3/2} \right] \\ &= \boxed{\frac{2}{27} \left[28^{3/2} - 19^{3/2} \right]} \end{aligned}$$

Math 181, Exam 2, Spring 2010
Problem 3a Solution

3a. Find the Maclaurin polynomial of degree 3 of the function $f(x) = \ln(x+1)$ centered at $a = 0$.

Solution: The 3rd degree Maclaurin polynomial $T_3(x)$ of $f(x)$ has the formula:

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

The derivatives of $f(x)$ and their values at $x = 0$ are:

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$\ln(x+1)$	$\ln(0+1) = 0$
1	$\frac{1}{x+1}$	$\frac{1}{0+1} = 1$
2	$-\frac{1}{(x+1)^2}$	$-\frac{1}{(0+1)^2} = -1$
3	$\frac{2}{(x+1)^3}$	$\frac{2}{(0+1)^3} = 2$

The function $T_3(x)$ is then:

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$T_3(x) = 0 + x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3$$

$$\boxed{T_3(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3}$$

Math 181, Exam 2, Spring 2010
Problem 3b Solution

3b. Find the Maclaurin polynomial of degree 3 of the function $f(x) = xe^x$ centered at $a = 0$.

Solution: The 3rd degree Maclaurin polynomial $T_3(x)$ of $f(x)$ has the formula:

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

The derivatives of $f(x)$ and their values at $x = 0$ are:

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	xe^x	$0 \cdot e^0 = 0$
1	$(x+1)e^x$	$(0+1)e^0 = 1$
2	$(x+2)e^x$	$(0+2)e^0 = 2$
3	$(x+3)e^x$	$(0+3)e^0 = 3$

The function $T_3(x)$ is then:

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$T_3(x) = 0 + x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3$$

$$\boxed{T_3(x) = x + x^2 + \frac{1}{2}x^3}$$

Math 181, Exam 2, Spring 2010
Problem 4a Solution

4a. Find the sum of the series:

$$\sum_{n=3}^{+\infty} \frac{3^{2n-1}}{5^{3n-2}}$$

Solution: We recognize the given series as a geometric series. In order to find its sum we must first rewrite the series.

$$\sum_{n=3}^{+\infty} \frac{3^{2n-1}}{5^{3n-2}} = \sum_{n=3}^{+\infty} \frac{3^{2n}3^{-1}}{5^{3n}5^{-2}} = \sum_{n=3}^{+\infty} \frac{3^{-1}}{5^{-2}} \cdot \frac{(3^2)^n}{(5^3)^n} = \sum_{n=3}^{+\infty} \frac{25}{3}\left(\frac{9}{125}\right)^n$$

This is a convergent geometric series because $|r| = |\frac{9}{125}| < 1$. We can now use the formula:

$$\sum_{n=M}^{+\infty} cr^n = r^M \cdot \frac{c}{1-r}$$

where $M = 3$, $c = \frac{25}{3}$, and $r = \frac{9}{125}$. The sum of the series is then:

$$\sum_{n=3}^{+\infty} \frac{25}{3}\left(\frac{9}{125}\right)^n = \left(\frac{9}{125}\right)^3 \cdot \frac{\frac{25}{3}}{1-\frac{9}{125}} = \boxed{\frac{243}{72,500}}$$

Math 181, Exam 2, Spring 2010
Problem 4b Solution

4b. Find the sum of the series: $\sum_{n=3}^{+\infty} \frac{3^{2n+1}}{4^{3n-2}}$

Solution: We recognize the given series as a geometric series. In order to find its sum we must first rewrite the series.

$$\sum_{n=3}^{+\infty} \frac{3^{2n+1}}{4^{3n-2}} = \sum_{n=3}^{+\infty} \frac{3^{2n}3^{1}}{4^{3n}4^{-2}} = \sum_{n=3}^{+\infty} \frac{3^1}{4^{-2}} \cdot \frac{(3^2)^n}{(4^3)^n} = \sum_{n=3}^{+\infty} 48 \left(\frac{9}{64}\right)^n$$

This is a convergent geometric series because $|r| = |\frac{9}{64}| < 1$. We can now use the formula:

$$\sum_{n=M}^{+\infty} cr^n = r^M \cdot \frac{c}{1-r}$$

where $M = 3$, $c = 48$, and $r = \frac{9}{64}$. The sum of the series is then:

$$\sum_{n=3}^{+\infty} 48 \left(\frac{9}{64}\right)^n = \left(\frac{9}{64}\right)^3 \cdot \frac{48}{1-\frac{9}{64}} = \boxed{\frac{2,187}{14,080}}$$

Math 181, Exam 2, Spring 2010
Problem 4c Solution

4c. Find the sum of the series: $\sum_{n=3}^{+\infty} \frac{2^{3n-1}}{3^{2n-2}}$

Solution: We recognize the given series as a geometric series. In order to find its sum we must first rewrite the series.

$$\sum_{n=3}^{+\infty} \frac{2^{3n-1}}{3^{2n-2}} = \sum_{n=3}^{+\infty} \frac{2^{3n}2^{-1}}{3^{2n}3^{-2}} = \sum_{n=3}^{+\infty} \frac{2^{-1}}{3^{-2}} \cdot \frac{(2^3)^n}{(3^2)^n} = \sum_{n=3}^{+\infty} \frac{9}{2}\left(\frac{8}{9}\right)^n$$

This is a convergent geometric series because $|r| = |\frac{8}{9}| < 1$. We can now use the formula:

$$\sum_{n=M}^{+\infty} cr^n = r^M \cdot \frac{c}{1-r}$$

where $M = 3$, $c = \frac{9}{2}$, and $r = \frac{8}{9}$. The sum of the series is then:

$$\sum_{n=3}^{+\infty} \frac{9}{2}\left(\frac{8}{9}\right)^n = \left(\frac{8}{9}\right)^3 \cdot \frac{\frac{9}{2}}{1-\frac{8}{9}} = \boxed{\frac{256}{9}}$$

Math 181, Exam 2, Spring 2010
Problem 5a Solution

5a. Determine whether the following series converge or not:

$$\sum_{n=2}^{+\infty} \frac{1}{n(\ln n)^3}, \qquad \sum_{n=1}^{+\infty} \frac{n}{\sqrt{n^5+n+1}}$$

Solution: We use the Integral Test to determine whether or not the first series converges. Let $f(x) = \frac{1}{x(\ln x)^3}$. The function $f(x)$ is decreasing for $x \geq 2$. We must now determine whether or not the following integral converges:

$$\int_2^{\infty} \frac{1}{x(\ln x)^3}\,dx = \lim_{R\to\infty} \int_2^{R} \frac{1}{x(\ln x)^3}\,dx$$

Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$ and we get:

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x(\ln x)^3}\,dx &= \lim_{R\to\infty} \int_2^{R} \frac{1}{x(\ln x)^3}\,dx \\
&= \lim_{R\to\infty} \int_{\ln 2}^{\ln R} \frac{1}{u^3}\,du \\
&= \lim_{R\to\infty} \left[-\frac{1}{2u^2}\right]_{\ln 2}^{\ln R} \\
&= \lim_{R\to\infty} \left(\frac{1}{2(\ln 2)^2} - \frac{1}{2(\ln R)^2}\right) \\
&= \frac{1}{2(\ln 2)^2}
\end{aligned}$$

Since the integral converges, the series $\sum_{n=2}^{+\infty} \frac{1}{n(\ln n)^3}$ **converges** by the Integral Test.

We use the Comparison Test to determine whether the second series converges or not. We guess that the series converges. Now let $a_n = \frac{n}{\sqrt{n^5+n+1}}$. We must find a series $\sum b_n$ such that (1) $0 \leq a_n \leq b_n$ for $n \geq 1$ and (2) $\sum b_n$ converges. We notice that:

$$0 \leq \frac{n}{\sqrt{n^5+n+1}} \leq \frac{n}{\sqrt{n^5}} = \frac{1}{n^{3/2}}$$

for all $n \geq 1$ using the argument that $\sqrt{n^5+n+1} > \sqrt{n^5}$ for $n \geq 1$. So we choose $b_n = \frac{1}{n^{3/2}}$. The series $\sum_{n=1}^{+\infty} \frac{1}{n^{3/2}}$ converges because it is a p-series with $p = \frac{3}{2} > 1$. Therefore, the series $\sum_{n=1}^{+\infty} \frac{n}{\sqrt{n^5+n+1}}$ **converges** by the Comparison Test.

Math 181, Exam 2, Spring 2010
Problem 5b Solution

5b. Determine whether the following series converge or not:

$$\sum_{n=2}^{+\infty} \frac{\ln n}{n^3}, \qquad \sum_{n=0}^{+\infty} \frac{(-1)^n}{\sqrt{n}+10}$$

Solution: We use the Comparison Test to determine whether the first series converges or not. We guess that the series converges. Let $a_n = \frac{\ln n}{n^3}$. We must choose a series $\sum b_n$ such that (1) $0 \le a_n \le b_n$ for $n \ge 2$ and (2) $\sum b_n$ converges. We notice that:

$$0 \le \frac{\ln n}{n^3} \le \frac{n}{n^3} = \frac{1}{n^2}$$

for $n \ge 2$. So we choose $b_n = \frac{1}{n^2}$. Since the series $\sum_{n=2}^{+\infty} \frac{1}{n^2}$ converges because it is a p-series with $p = 2 > 1$, the series $\sum_{n=2}^{+\infty} \frac{\ln n}{n^3}$ **converges** by the Comparison Test.

The second series is alternating so we use the Leibniz Test to determine if it converges or not. Let $a_n = \frac{1}{\sqrt{n}+10}$. We know that a_n is decreasing for $n > 0$. Furthermore,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}+10} = 0$$

Therefore, the series $\sum_{n=0}^{+\infty} \frac{(-1)^n}{\sqrt{n}+10}$ **converges** by the Leibniz Test.

Math 181, Exam 2, Spring 2010
Problem 6a Solution

6a. Find the volume of the solid obtained by rotating the region below the graph of $y = \frac{1}{x+1}$ about the x-axis for $0 \leq x < \infty$.

Solution: The volume of the solid is obtained using the Disk Method. The formula we will use is:

$$V = \pi \int_0^\infty \left(\frac{1}{x+1} \right)^2 dx$$

To compute the integral we use the u-substitution method letting $u = x+1$, $du = dx$. Using the equation $u = x+1$, we see that the lower limit of integration changes from 0 to 1 but the upper limit is still ∞. The integral becomes:

$$\begin{aligned} V &= \pi \int_0^\infty \left(\frac{1}{x+1} \right)^2 dx \\ &= \pi \int_0^\infty \frac{1}{(x+1)^2}\, dx \\ &= \pi \int_1^\infty \frac{1}{u^2}\, du \end{aligned}$$

This is a p-integral with $p = 2 > 1$ so we know that the integral converges. In fact, its value is:

$$V = \pi \int_1^\infty \frac{1}{u^2}\, du = \pi \cdot \frac{1}{2-1} = \boxed{\pi}$$

Math 181, Exam 2, Spring 2010
Problem 6b Solution

6b. Find the volume of the solid obtained by rotating the region below the graph of $y = \frac{1}{x^2+1}$ about the x-axis for $0 \le x < \infty$.

Solution: The volume of the solid is obtained using the Disk Method. The formula we will use is:

$$V = \pi \int_0^\infty \left(\frac{1}{x^2+1}\right)^2 dx = \pi \int_0^\infty \frac{1}{(x^2+1)^2}\,dx$$

To compute the integral we first turn it into a limit calculation.

$$V = \pi \int_0^\infty \frac{1}{(x^2+1)^2}\,dx = \lim_{R\to\infty} \pi \int_0^\infty \frac{1}{(x^2+1)^2}\,dx$$

We use the trigonometric substitution method to evaluate the integral letting $x = \tan\theta$ and $dx = \sec^2\theta\,d\theta$. The indefinite integral becomes:

$$\begin{aligned}\int \frac{1}{(x^2+1)^2}\,dx &= \int \frac{1}{(\tan^2\theta+1)^2}\left(\sec^2\theta\,d\theta\right)\\ &= \int \frac{1}{(\sec^2\theta)^2}\left(\sec^2\theta\,d\theta\right)\\ &= \int \cos^2\theta\,d\theta\\ &= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\end{aligned}$$

Using the fact that $x = \tan\theta$ we find that $\theta = \arctan x$, $\sin\theta = \frac{x}{\sqrt{x^2+1}}$, and $\cos\theta = \frac{1}{\sqrt{x^2+1}}$ either using a triangle or a few Pythagorean identities. The indefinite integral in terms of x is:

$$\begin{aligned}\int \frac{1}{(x^2+1)^2}\,dx &= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\\ &= \frac{1}{2}\arctan x + \frac{1}{2}\left(\frac{x}{\sqrt{x^2+1}}\right)\left(\frac{1}{\sqrt{x^2+1}}\right)\\ &= \frac{1}{2}\arctan x + \frac{x}{2(x^2+1)}\end{aligned}$$

The definite integral from 0 to R is:

$$\begin{aligned}\int_0^R \frac{1}{(x^2+1)^2}\,dx &= \left[\frac{1}{2}\arctan x + \frac{x}{2(x^2+1)}\right]_0^R\\ &= \left[\frac{1}{2}\arctan R + \frac{R}{2(R^2+1)}\right] - \left[\frac{1}{2}\arctan 0 + \frac{0}{2(0^2+1)}\right]\\ &= \frac{1}{2}\arctan R + \frac{R}{2(R^2+1)}\end{aligned}$$

The volume is then:

$$
\begin{aligned}
V &= \pi \int_0^{\infty} \frac{1}{(x^2+1)^2}\,dx \\
&= \lim_{R\to\infty} \pi \int_0^{\infty} \frac{1}{(x^2+1)^2}\,dx \\
&= \lim_{R\to\infty} \pi \left[\frac{1}{2}\arctan R + \frac{R}{2(R^2+1)}\right] \\
&= \frac{\pi}{2} \lim_{R\to\infty} \arctan R + \frac{\pi}{2} \lim_{R\to\infty} \frac{R}{R^2+1} \\
&\overset{\text{L'H}}{=} \frac{\pi}{2}\cdot\frac{\pi}{2} + \frac{\pi}{2} \lim_{R\to\infty} \frac{(R)'}{(R^2+1)'} \\
&= \frac{\pi}{2}\cdot\frac{\pi}{2} + \frac{\pi}{2} \lim_{R\to\infty} \frac{1}{2R} \\
&= \frac{\pi}{2}\cdot\frac{\pi}{2} + \frac{\pi}{2}\cdot 0 \\
&= \boxed{\frac{\pi^2}{4}}
\end{aligned}
$$