Math 181, Exam 2, Spring 2012
Problem 1 Solution

1. Compute the sums of the following series (do not show that they converge).

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{2^{k-1}}{3^{2k}}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{2}{(n-1)(n+1)}$

Solution:

(a) We begin by rewriting the series as follows:

$$\sum_{k=0}^{\infty} \frac{2^{k-1}}{3^{2k}} = \sum_{k=0}^{\infty} \frac{2^k \cdot 2^{-1}}{(3^2)^k} = \sum_{k=0}^{\infty} \frac{2^k \cdot \frac{1}{2}}{9^k} = \sum_{k=0}^{\infty} \frac{1}{2}\left(\frac{2}{9}\right)^k.$$

Recognizing that this is a geometric series with $a = \frac{1}{2}$ and $r = \frac{2}{9}$ we know that the series converges because $|r| < 1$ and that its sum is

$$\sum_{k=0}^{\infty} \frac{1}{2}\left(\frac{2}{9}\right)^k = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{2}{9}} = \frac{9}{14}.$$

(b) This is a telescoping series. The partial fraction decomposition of the nth term is

$$\frac{2}{(n-1)(n+1)} = \frac{1}{n-1} - \frac{1}{n+1}.$$

The Nth partial sum of the series is

$$S_N = \sum_{n=2}^{N} \left(\frac{1}{n-1} - \frac{1}{n+1}\right),$$

$$S_N = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots$$
$$+ \left(\frac{1}{N-4} - \frac{1}{N-2}\right) + \left(\frac{1}{N-3} - \frac{1}{N-1}\right) + \left(\frac{1}{N-2} - \frac{1}{N}\right) + \left(\frac{1}{N-1} - \frac{1}{N+1}\right),$$

$$S_N = 1 + \frac{1}{2} - \frac{1}{N} - \frac{1}{N+1}.$$

The sum of the series is the limit of S_N as $N \to \infty$. That is,

$$\sum_{n=2}^{\infty} \frac{2}{(n-1)(n+1)} = \lim_{N\to\infty} S_N,$$

$$\sum_{n=2}^{\infty} \frac{2}{(n-1)(n+1)} = \lim_{N\to\infty} \left(1 + \frac{1}{2} - \frac{1}{N} - \frac{1}{N+1}\right),$$

$$\sum_{n=2}^{\infty} \frac{2}{(n-1)(n+1)} = 1 + \frac{1}{2} - 0 - 0$$

$$\sum_{n=2}^{\infty} \frac{2}{(n-1)(n+1)} = \frac{3}{2}$$

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Problem 2 Solution

2. For each sequence below, determine its limit or show that it diverges.

(a) $\left\{ \dfrac{5^{2n}}{2^{5n}} \right\}$

(b) $\{(2n)^{1/n}\}$

Solution:

(a) We begin by rewriting the nth term of the sequence as follows:

$$\frac{5^{2n}}{2^{5n}} = \frac{(5^2)^n}{(2^5)^n} = \frac{25^n}{32^n} = \left(\frac{25}{32}\right)^n$$

The sequence is geometric with $|r| = |\frac{25}{32}| < 1$. Therefore, we know that it converges to 0. That is,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{25}{32}\right)^n = 0$$

(b) First, we notice that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (2n)^{1/n} \to \infty^0$$

which is indeterminate. We resolve this indeterminacy by rewriting the function using the exponential of a logarithm. That is,

$$(2n)^{1/n} = e^{\ln(2n)^{1/n}} = e^{\frac{1}{n}\ln(2n)}$$

Therefore, the value of the limit is

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} e^{\frac{1}{n}\ln(2n)}, \\
\lim_{n\to\infty} a_n &= e^{\lim_{n\to\infty} \frac{\ln(2n)}{n}}, \\
\lim_{n\to\infty} a_n &= e^0, \\
\lim_{n\to\infty} a_n &= 1.
\end{aligned}$$

We used the fact that $\ln(2n) << n$ as $n \to \infty$ to evaluate the limit.

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Problem 3 Solution

3. Compute the integral or show that it diverges.

(a) $\displaystyle\int_2^\infty \frac{dx}{x^2+4}$

(b) $\displaystyle\int_2^3 \frac{dx}{(x-2)^{5/4}}$

Solution:

(a) We begin by rewriting the integral as a limit.

$$\int_2^\infty \frac{dx}{x^2+4} = \lim_{b\to\infty} \int_2^b \frac{dx}{x^2+4}$$

Using the trigonometric substitution $x = 2\tan(\theta)$ one can show that an antiderivative of $\frac{1}{x^2+4}$ is

$$\int \frac{dx}{x^2+4} = \frac{1}{2}\arctan\left(\frac{x}{2}\right)$$

Therefore, the value of the integral is

$$\begin{aligned}
\int_2^\infty \frac{dx}{x^2+4} &= \lim_{b\to\infty} \int_2^b \frac{dx}{x^2+4}, \\
\int_2^\infty \frac{dx}{x^2+4} &= \lim_{b\to\infty} \left[\frac{1}{2}\arctan\left(\frac{x}{2}\right)\right]_2^b \\
\int_2^\infty \frac{dx}{x^2+4} &= \lim_{b\to\infty} \left[\frac{1}{2}\arctan\left(\frac{b}{2}\right) - \frac{1}{2}\arctan(1)\right], \\
\int_2^\infty \frac{dx}{x^2+4} &= \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\cdot\frac{\pi}{4} \\
\int_2^\infty \frac{dx}{x^2+4} &= \frac{\pi}{8}.
\end{aligned}$$

(b) We begin by letting $u = x - 2$ and $du = dx$. The limits of integration then become $u = 0$ and $u = 1$ upon substituting the original limits into the equation $u = x - 2$. Therefore, the integral becomes

$$\int_2^3 \frac{dx}{(x-2)^{5/4}} = \int_0^1 \frac{du}{u^{5/4}}$$

which we recognize as a p-integral with $p = \frac{5}{4}$. Since $p > 1$ we know that the integral diverges.

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Problem 4 Solution

4. Use the Trapezoid Rule with 3 subintervals to approximate $\int_0^{\pi} \sin(x)\,dx$.

Solution: Since $a = 0$, $b = \pi$, and $N = 3$ we know that

$$\Delta x = \frac{b-a}{N} = \frac{\pi - 0}{3} = \frac{\pi}{3}.$$

The Trapezoidal estimate is then

$$T_3 = \frac{\Delta x}{2}\left[f(0) + 2f\left(\frac{\pi}{3}\right) + 2f\left(\frac{2\pi}{3}\right) + f(\pi)\right],$$

$$T_3 = \frac{\frac{\pi}{3}}{2}\left[\sin(0) + 2\sin\left(\frac{\pi}{3}\right) + 2\sin\left(\frac{2\pi}{3}\right) + \sin(\pi)\right],$$

$$T_3 = \frac{\pi}{6}\left[0 + 2\left(\frac{\sqrt{3}}{2}\right) + 2\left(\frac{\sqrt{3}}{2}\right) + 0\right],$$

$$T_3 = \frac{\pi\sqrt{3}}{3}.$$

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Problem 5 Solution

5. Determine whether each of the following series converges or diverges. Indicate the method you are using.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{k^4+1}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{k^k}$

Solution:

(a) First, we note that

$$0 \leq \frac{k^2}{k^4+1} \leq \frac{k^2}{k^4} = \frac{1}{k^2}$$

for all k. Furthermore, the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series since $p = 2 > 1$. Thus, we know that $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{k^4+1}$ converges by the Comparison Test.

(b) Due to the presence of the factorial, we know that the Ratio Test is the preferred convergence test. The value of r, the limit of the ratio of consecutive terms as $k \to \infty$, is

$$\begin{aligned}
r &= \lim_{k\to\infty} \frac{a_{k+1}}{a_k}, \\
r &= \lim_{k\to\infty} \frac{(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{k!}, \\
r &= \lim_{k\to\infty} \frac{(k+1)k!}{(k+1)^k(k+1)} \cdot \frac{k^l}{k!}, \\
r &= \lim_{k\to\infty} \frac{k^k}{(k+1)^k}, \\
r &= \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k, \\
r &= \lim_{k\to\infty} \frac{1}{\left(\frac{k+1}{k}\right)^k}, \\
r &= \lim_{k\to\infty} \frac{1}{\left(1+\frac{1}{k}\right)^k}, \\
r &= \frac{1}{\lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k}, \\
r &= \frac{1}{e}.
\end{aligned}$$

Therefore, since $r = \frac{1}{e} < 1$ we know that the series converges.