Math 181, Exam 2, Spring 2013
Problem 1 Solution

1. Compute the following sums

(a) $\displaystyle\sum_{n=3}^{+\infty} \frac{2}{n(n-1)}$

(b) $\displaystyle\sum_{n=1}^{+\infty} \frac{2^{n+3}}{5 \cdot 7^{3n-2}}$

Solution:

(a) This is a **telescoping series**. To compute the sum, we decompose the summand as follows:

$$\frac{2}{n(n-1)} = \frac{2}{n-1} - \frac{2}{n}$$

The Nth partial sum of the series is

$$S_N = \left(\frac{2}{2} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{2}{4}\right) + \cdots + \left(\frac{2}{N-2} - \frac{2}{N-1}\right) + \left(\frac{2}{N-1} - \frac{2}{N}\right)$$

The sum collapses into the following:

$$S_N = \frac{2}{2} - \frac{2}{N}$$

The sum of the series is the limit of S_N as $N \to \infty$. That is,

$$\sum_{n=3}^{+\infty} \frac{2}{n(n-1)} = \lim_{N\to\infty} S_N$$

$$\sum_{n=3}^{+\infty} \frac{2}{n(n-1)} = \lim_{N\to\infty} \left(\frac{2}{2} - \frac{2}{N}\right)$$

$$\boxed{\sum_{n=3}^{+\infty} \frac{2}{n(n-1)} = 1}$$

(b) This is a **geometric series**. To begin, we rewrite the summand as follows:

$$\frac{2^{n+3}}{5 \cdot 7^{3n-2}} = \frac{2^n \cdot 2^3}{5 \cdot 7^{3n} \cdot 7^{-2}} = \frac{2^3}{5 \cdot 7^{-2}} \cdot \frac{2^n}{(7^3)^n} = \frac{8}{5 \cdot 49^{-1}} \cdot \left(\frac{2}{7^3}\right)^n = \frac{392}{5} \cdot \left(\frac{2}{343}\right)^n$$

Using the fact that

$$\sum_{n=N}^{\infty} ar^n = r^N \cdot \frac{a}{1-r}, \quad |r| < 1$$

we have

$$\sum_{n=1}^{+\infty} \frac{2^{n+3}}{5 \cdot 7^{3n-2}} = \left(\frac{2}{343}\right)^1 \cdot \frac{\frac{392}{5}}{1 - \frac{2}{343}}$$

$$\sum_{n=1}^{+\infty} \frac{2^{n+3}}{5 \cdot 7^{3n-2}} = \frac{2}{343} \cdot \frac{\frac{392}{5}}{\frac{341}{343}}$$

$$\boxed{\sum_{n=1}^{+\infty} \frac{2^{n+3}}{5 \cdot 7^{3n-2}} = \frac{784}{1705}}$$

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Problem 2 Solution

2. Compute the integral $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{x^2+4x+5}$.

Solution: We begin by splitting the integral as follows:

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+4x+5} = \int_{-\infty}^{-2} \frac{dx}{x^2+4x+5} + \int_{-2}^{\infty} \frac{dx}{x^2+4x+5}$$

We split the integral at -2 because the denominator becomes $(x+2)^2+1$ after completing the square. Letting $u = x+2$, $du = dx$ then gives us the sum

$$\int_{-\infty}^{0} \frac{du}{u^2+1} + \int_{0}^{\infty} \frac{du}{u^2+1}$$

Each integral has the same value due to the function $f(u) = \frac{1}{u^2+1}$ being even, i.e. it has symmetry with respect to the y-axis. The second integral evaluates to

$$\begin{aligned} \int_{0}^{\infty} \frac{du}{u^2+1} &= \lim_{b\to\infty} \int_{0}^{b} \frac{du}{u^2+1} \\ &= \lim_{b\to\infty} \left[\tan^{-1}(b) - \tan^{-1}(0)\right] \\ &= \frac{\pi}{2} \end{aligned}$$

Thus, the value of the improper integral is

$$\boxed{\int_{-\infty}^{\infty} \frac{dx}{x^2+4x+5} = \frac{\pi}{2} + \frac{\pi}{2} = \pi}$$

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Problem 3 Solution

3. Determine whether the following series converge or not.

(a) $\displaystyle\sum_{n=1}^{+\infty} \cos\left(\frac{1}{n}\right)$

(b) $\displaystyle\sum_{n=1}^{+\infty} \left(\frac{n}{5n+3}\right)^n$

(c) $\displaystyle\sum_{n=1}^{+\infty} \frac{\sin^2(n)}{n^2}$

Solution:

(a) Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1 \neq 0$, the series diverges by the **Divergence Test**.

(b) Since $\rho = \lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{n}{5n+3} = \frac{1}{5} < 1$ the series converges by the **Root Test.**

(c) Using the fact that $0 \leq \sin^2(n) \leq 1$ for all n we have

$$0 \leq \frac{\sin^2(n)}{n^2} \leq \frac{1}{n^2}$$

for all $n \geq 1$ and that $\sum \frac{1}{n^2}$ is a convergent p-series, we can say that the series $\sum \frac{\sin^2(n)}{n^2}$ converges by the **Comparison Test**.

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Problem 4 Solution

4. Determine whether the following integrals converge or not:

(a) $\displaystyle\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$

(b) $\displaystyle\int_2^{+\infty} \frac{1}{x(\ln x)^2}\,dx$

Solution:

(a) Letting $u = 4 - x$, $du = -dx$ the integral transforms as follows:

$$\begin{aligned}\int_0^4 \frac{1}{\sqrt{4-x}}\,dx &= -\int_4^0 \frac{1}{\sqrt{u}}\,du \\ &= \int_0^4 \frac{1}{\sqrt{u}}\,du \\ &= \int_0^1 \frac{du}{\sqrt{u}} + \int_1^4 \frac{du}{\sqrt{u}}\end{aligned}$$

where the first integral on the right hand side above is known to be a convergent p-integral and has the value

$$\int_0^1 \frac{du}{u^{1/2}} = \frac{1}{1-\frac{1}{2}} = 2$$

The second integral is proper and has the value

$$\int_1^4 \frac{du}{\sqrt{u}} = 2\sqrt{u}\Big|_1^4 = 2\sqrt{4} - 2\sqrt{1} = 2$$

Thus, the improper integral converges and has the value

$$\boxed{\int_0^4 \frac{dx}{\sqrt{4-x}} = 2 + 2 = 4}$$

(b) Letting $u = \ln(x)$ and $du = \frac{1}{x}\,dx$ the integral transforms as follows:

$$\begin{aligned}\int_2^{+\infty} \frac{dx}{x(\ln x)^2} &= \int_{\ln(2)}^{+\infty} \frac{du}{u^2} \\ &= \int_{\ln(2)}^{1} \frac{du}{u^2} + \int_1^{+\infty} \frac{du}{u^2}\end{aligned}$$

The second integral is a p-integral whose value is

$$\int_{1}^{+\infty} \frac{du}{u^2} = \frac{1}{2-1} = 1$$

The first integral is evaluated as follows

$$\int_{\ln(2)}^{1} \frac{du}{u^2} = \left[-\frac{1}{u}\right]_{\ln(2)}^{1} = -1 + \frac{1}{\ln(2)}$$

Thus, the improper integral converges and has the value

$$\boxed{\int_{2}^{+\infty} \frac{dx}{x(\ln x)^2} = 1 - 1 + \frac{1}{\ln(2)} = \frac{1}{\ln(2)}}$$

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Problem 5 Solution

5. Compute the limit of each sequence or show that the sequence diverges.

(a) $\{a_n\} = \left\{\sqrt[n]{n^2+n+3}\right\}$

(b) $\{b_n\} = \left\{\dfrac{n+\sin n}{2n-\cos n}\right\}$

Solution:

(a) To begin, we rewrite the function as

$$\sqrt[n]{n^2+n+3} = (n^2+n+3)^{1/n} = \exp\left(\ln(n^2+n+3)^{1/n}\right)$$

where $\exp(x) = e^x$, by definition. Using the logarithm rule $\ln(x^n) = n\ln(x)$ we have

$$\exp\left(\frac{1}{n}\ln(n^2+n+3)\right) = \exp\left(\frac{\ln(n^2+n+3)}{n}\right)$$

We now use Theorems ? and ? to find the value of the limit. That is,

$$\begin{aligned}\lim_{n\to\infty}(n^2+n+3)^{1/n} &= \exp\left(\lim_{x\to\infty}\frac{\ln(x^2+x+3)}{x}\right)\\ &= \exp\left(\lim_{x\to\infty}\frac{2x+1}{x^2+x+3}\right)\\ &= \exp\left(\lim_{x\to\infty}\frac{2}{x}\right)\\ &= \exp(0)\\ &= 1\end{aligned}$$

(b) The Squeeze Theorem is appropriate here. We know that

$$\frac{n-1}{2n+1} \le \frac{n+\sin(n)}{2n-\cos(n)} \le \frac{n+1}{2n-1}$$

for all $n \ge 0$ and that

$$\lim_{n\to\infty}\frac{n-1}{2n+1} = \lim_{n\to\infty}\frac{n+1}{2n-1} = \frac{1}{2}$$

Thus, the sequence converges to $\frac{1}{2}$.