Math 181, Exam 2, Study Guide 2
Problem 1 Solution

1. Use the trapezoid rule with $n = 2$ to estimate the arc-length of the curve $y = \sin x$ between $x = 0$ and $x = \pi$.

Solution: The arclength is:

$$\begin{aligned} L &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_0^\pi \sqrt{1 + (\cos x)^2}\, dx \\ &= \int_0^\pi \sqrt{1 + \cos^2 x}\, dx \end{aligned}$$

We now use the trapezoid rule with $n = 2$ to estimate the value of the integral. The formula we will use is:

$$T_2 = \frac{\Delta x}{2}\left[f(0) + 2f\left(\frac{\pi}{2}\right) + f(\pi)\right]$$

where $f(x) = \sqrt{1 + \cos^2 x}$ and the value of Δx is:

$$\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{2} = \frac{\pi}{2}$$

The value of T_2 is then:

$$\begin{aligned} T_2 &= \frac{\Delta x}{2}\left[f(0) + 2f\left(\frac{\pi}{2}\right) + f(\pi)\right] \\ &= \frac{\frac{\pi}{2}}{2}\left[\sqrt{1 + \cos^2 0} + 2\sqrt{1 + \cos^2 \frac{\pi}{2}} + \sqrt{1 + \cos^2 \pi}\right] \\ &= \frac{\pi}{4}\left[\sqrt{1+1} + 2\sqrt{1+0} + \sqrt{1+1}\right] \\ &= \boxed{\frac{\pi}{4}\left(2 + \sqrt{2}\right)} \end{aligned}$$

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Problem 2 Solution

2.

(a) Let R be the region between $y = \frac{1}{1+x^2}$ and the x-axis with $x \geq 0$. Does R have finite area? If so, what is the area?

(b) Let S be the solid obtained by revolving R around the y-axis. Does S have finite volume? If so, what is the volume?

Solution:

(a) The area of R is given by the improper integral:

$$\text{Area} = \int_0^{+\infty} \frac{1}{x^2+1}\,dx$$

We evaluate the integral by turning it into a limit calculation.

$$\int_0^{+\infty} \frac{dx}{x^2+1} = \lim_{R\to+\infty} \int_0^R \frac{dx}{x^2+1}$$

The integral has a simple antiderivative so its value is:

$$\begin{aligned} \int_0^R \frac{dx}{x^2+1} &= \Big[\arctan x\Big]_0^R \\ &= \arctan R - \arctan 0 \\ &= \arctan R \end{aligned}$$

We now take the limit of the above function as $R \to +\infty$.

$$\begin{aligned} \int_0^{+\infty} \frac{dx}{x^2+1} &= \lim_{R\to+\infty} \int_0^R \frac{dx}{x^2+1} \\ &= \lim_{R\to+\infty} \arctan R \\ &= \frac{\pi}{2} \end{aligned}$$

Thus, the area is finite and its value is $\boxed{\frac{\pi}{2}}$.

(b) The volume of S is obtained by using the Shell Method. The formula is

$$V = \int_0^{\infty} 2\pi x \cdot \frac{1}{x^2+1}\,dx$$

To compute the integral we first turn it into a limit calculation.

$$V = \pi \lim_{b\to\infty} \int_0^b \frac{2x}{x^2+1}\,dx$$

The value of the integral is

$$\int_0^b \frac{2x}{x^2+1}\,dx = \left[\ln|x^2+1|\right]_0^b = \ln(b^2+1)$$

The volume is then

$$V = \lim_{b\to\infty}, \ln(b^2+1) = \infty$$

That is, the volume is not finite.

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Problem 3 Solution

3. Evaluate the following integrals:

(a) $\displaystyle\int_{-\pi}^{\pi} \sin^4 x\, dx$

(b) $\displaystyle\int_0^1 \frac{dx}{2x^2+5x+2}$

(c) $\displaystyle\int_0^1 \frac{dx}{2x^2+4x+3}$

(d) $\displaystyle\int_0^{\infty} x^2 e^{-x}\, dx$

Solution:

(a) We solve this integral using a reduction formula.

$$\int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx$$

Letting $n=4$ we get:

$$\begin{aligned}\int \sin^4 x\, dx &= -\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\int \sin^2 x\, dx \\ &= -\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\int \frac{1}{2}\left[1-\cos(2x)\right]\, dx \\ &= -\frac{1}{4}\sin^3 x\cos x + \frac{3}{8}\int \left[1-\cos(2x)\right]\, dx \\ &= -\frac{1}{4}\sin^3 x\cos x + \frac{3}{8}x - \frac{3}{16}\sin(2x)\end{aligned}$$

To evaluate $\displaystyle\int \sin^2 x\, dx$ we used the double angle identity $\cos(2x) = 1 - 2\sin^2 x$.

We now solve the definite integral.

$$\begin{aligned}\int_{-\pi}^{\pi} \sin^4 x\, dx &= \left[-\frac{1}{4}\sin^3 x\cos x + \frac{3}{8}x - \frac{3}{16}\sin(2x)\right]_{-\pi}^{\pi} \\ &= \left[-\frac{1}{4}\sin^3 \pi\cos \pi + \frac{3}{8}\pi - \frac{3}{16}\sin(2\pi)\right] - \left[-\frac{1}{4}\sin^3(-\pi)\cos(-\pi) + \frac{3}{8}(-\pi) - \frac{3}{16}\sin(-2\pi)\right] \\ &= \left[0 + \frac{3}{8}\pi - 0\right] - \left[0 - \frac{3}{8}\pi - 0\right] \\ &= \boxed{\frac{3}{4}\pi}\end{aligned}$$

(b) We will evaluate the integral using Partial Fraction Decomposition. First, we factor the denominator and then decompose the rational function into a sum of simpler rational functions.

$$\frac{1}{2x^2+5x+2} = \frac{1}{(2x+1)(x+2)} = \frac{A}{2x+1} + \frac{B}{x+2}$$

Next, we multiply the above equation by $(2x+1)(x+2)$ to get:

$$1 = A(x+2) + B(2x+1)$$

Then we plug in two different values for x to create a system of two equations in two unknowns (A, B). We select $x = -\frac{1}{2}$ and $x = -2$ for simplicity.

$$x = -\frac{1}{2}: \;\; A\left(-\frac{1}{2}+2\right) + B\left(2\left(-\frac{1}{2}\right)+1\right) = 1 \quad \Rightarrow \quad A = \frac{2}{3}$$
$$x = -2: \;\; A(-2+2) + B(2(-2)+1) = 1 \quad \Rightarrow \quad B = -\frac{1}{3}$$

Finally, we plug these values for A and B back into the decomposition and integrate.

$$\begin{aligned}\int \frac{1}{2x^2+5x+2}\,dx &= \int \left(\frac{A}{2x+1} + \frac{B}{x+2}\right) dx \\ &= \int \left(\frac{\frac{2}{3}}{2x+1} + \frac{-\frac{1}{3}}{x+2}\right) dx \\ &= \frac{1}{3}\ln|2x+1| - \frac{1}{3}\ln|x+2|\end{aligned}$$

We now solve the definite integral.

$$\begin{aligned}\int_0^1 \frac{dx}{2x^2+5x+2} &= \left[\frac{1}{3}\ln|2x+1| - \frac{1}{3}\ln|x+2|\right]_0^1 \\ &= \left[\frac{1}{3}\ln|2(1)+1| - \frac{1}{3}\ln|1+2|\right] - \left[\frac{1}{3}\ln|2(0)+1| - \frac{1}{3}\ln|0+2|\right] \\ &= \frac{1}{3}\ln 3 - \frac{1}{3}\ln 3 - \frac{1}{3}\ln 1 + \frac{1}{3}\ln 2 \\ &= \boxed{\frac{1}{3}\ln 2}\end{aligned}$$

(c) We begin by completing the square in the denominator.

$$\int \frac{dx}{2x^2+4x+3} = \int \frac{dx}{2(x+1)^2+1}$$

We then evaluate the integral using the u-substitution method. Let $u = \sqrt{2}(x+1)$. Then $du = \sqrt{2}\,dx \quad \Rightarrow \quad \frac{1}{\sqrt{2}}\,du = dx$ and we get:

$$\begin{aligned}
\int \frac{dx}{2x^2+4x+3} &= \int \frac{dx}{2(x+1)^2+1} \\
&= \int \frac{dx}{[\sqrt{2}(x+1)]^2+1} \\
&= \int \frac{\frac{1}{\sqrt{2}}\,du}{u^2+1} \\
&= \frac{1}{\sqrt{2}} \int \frac{du}{u^2+1} \\
&= \frac{1}{\sqrt{2}} \arctan u + C \\
&= \frac{1}{\sqrt{2}} \arctan\left[\sqrt{2}(x+1)\right]
\end{aligned}$$

We now solve the definite integral.

$$\begin{aligned}
\int_0^1 \frac{dx}{2x^2+4x+3} &= \left[\frac{1}{\sqrt{2}} \arctan\left[\sqrt{2}(x+1)\right]\right]_0^1 \\
&= \frac{1}{\sqrt{2}} \arctan\left[\sqrt{2}(1+1)\right] - \frac{1}{\sqrt{2}} \arctan\left[\sqrt{2}(0+1)\right] \\
&= \boxed{\frac{1}{\sqrt{2}}\left[\arctan\left(2\sqrt{2}\right) - \arctan\left(\sqrt{2}\right)\right]}
\end{aligned}$$

(d) We evaluate the integral using Integration by Parts. Let $u = x^2$ and $v' = e^{-x}$. Then $u' = 2x$ and $v = -e^{-x}$. Using the Integration by Parts formula:

$$\int uv'\,dx = uv - \int u'v\,dx$$

we get:

$$\begin{aligned}
\int x^2 e^{-x}\,dx &= -x^2e^{-x} - \int 2x\left(-e^{-x}\right)\,dx \\
&= -x^2e^{-x} + 2\int xe^{-x}\,dx
\end{aligned}$$

A second Integration by Parts must be performed. Let $u = x$ and $v' = e^{-x}$. Then $u' = 1$ and $v = -e^{-x}$. Using the Integration by Parts formula again we get:

$$\begin{aligned}
\int x^2e^{-x}\,dx &= -x^2e^{-x} + 2\left[-xe^{-x} - \int \left(-e^{-x}\right)\,dx\right] \\
&= -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x}\,dx \\
&= -x^2e^{-x} - 2xe^{-x} - 2e^{-x}
\end{aligned}$$

We now solve the definite integral. We recognize that it is an improper integral so we turn it into a limit and evaluate.

$$\begin{aligned}
\int_0^\infty x^2 e^{-x}\,dx &= \lim_{b\to\infty} \int_0^b x^2 e^{-x}\,dx \\
&= \lim_{b\to\infty} \left[-x^2 e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^b \\
&= \lim_{b\to\infty} \left[-b^2 e^{-b} - 2be^{-b} - 2e^{-b} + 2\right] \\
&= \lim_{b\to\infty} \left[-\frac{b^2}{e^b} - \frac{2b}{e^b} - \frac{2}{e^b} + 2\right] \\
&= -0 - 0 - 0 + 2 \\
&= \boxed{2}
\end{aligned}$$

When computing the limits above, we used the fact that:

$$\lim_{x\to 0} \frac{x^n}{e^x} = 0$$

by repeated application of L'Hopital's Rule.

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Problem 4 Solution

4. Use a Taylor polynomial for $y = e^x$ to calculate e to two decimal places. Explain (using the remainder formula) why you have used sufficiently many terms.

Solution: We will find the nth degree Maclaurin polynomial of $f(x) = e^x$ so that the error $|T_n(1) - f(1)| = |T_n(1) - e|$ is less than 10^{-2}. That is, we must find a value of n that ensures that the Error Bound satisfies the inequality:

$$\text{Error} = |T_n(1) - e| \leq K\frac{|x-a|^{n+1}}{(n+1)!} < 10^{-2}$$

where $x = 1$, $a = 0$, and K satisfies the inequality $|f^{(n+1)}(u)| \leq K$ for all $u \in [0,1]$. Since $|f^{(n+1)}(u)| = e^u < 3$ for all $u \in [0,1]$ we choose $K = 3$. We now want to satisfy the inequality:

$$\text{Error} = |T_n(1) - e| \leq 3\frac{|1-0|^{n+1}}{(n+1)!} < 10^{-2}$$

$$\text{Error} = |T_n(1) - e| \leq \frac{3}{(n+1)!} < 10^{-2} = \frac{1}{100}$$

We will find an appropriate value of n by a trial and error process.

n	$\frac{3}{(n+1)!}$
1	$\frac{3}{2!} = \frac{3}{2}$
2	$\frac{3}{3!} = \frac{1}{6}$
3	$\frac{3}{4!} = \frac{1}{8}$
4	$\frac{3}{5!} = \frac{1}{40}$
5	$\frac{3}{6!} = \frac{1}{240} < \frac{1}{100}$

Therefore, we choose $\boxed{n = 5}$. The Maclaurin polynomial $T_5(x)$ for $f(x) = e^x$ is:

$$T_5(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5$$

Evaluating at $x = 1$ we get:

$$T_5(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}$$

$$T_5(1) = \frac{163}{60} \approx \boxed{2.71\bar{6}}$$

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Problem 5 Solution

5. Let S be the surface obtained by revolving the curve $y = \sin x$ between $x = 0$ and $x = \pi$ around the x-axis. What is the surface area of S?

Solution: The surface area formula is:

$$\text{Surface Area} = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx$$

Using $a = 0$, $b = \pi$, $f(x) = \sin x$, and $f'(x) = \cos x$ we get:

$$\begin{aligned}
\text{Surface Area} &= 2\pi \int_0^\pi \sin x\sqrt{1 + (\cos x)^2}\, dx \\
&= 2\pi \int_0^\pi \sin x\sqrt{1 + \cos^2 x}\, dx
\end{aligned}$$

To evaluate the integral we use the u-substitution $u = \cos x$. Then $-du = \sin x\, dx$, the lower limit of integration changes from 0 to 1, and the upper limit changes from π to -1. Making these substitutions we get:

$$\begin{aligned}
\text{Surface Area} &= -2\pi \int_1^{-1} \sqrt{1 + u^2}\, du \\
&= 2\pi \int_{-1}^{1} \sqrt{1 + u^2}\, du \\
&= 4\pi \int_0^1 \sqrt{1 + u^2}\, du
\end{aligned}$$

where, in the last step, we used the fact that $\sqrt{1 + u^2}$ is symmetric with respect to the y-axis. To evaluate this integral we'll use the trigonometric substitution $u = \tan\theta$ and $du = \sec^2\, d\theta$. We get:

$$\begin{aligned}
\int \sqrt{1 + u^2}\, du &= \int \sqrt{1 + \tan^2\theta}\, \sec^2\theta\, d\theta \\
&= \int \sqrt{\sec^2\theta}\, \sec^2\theta\, d\theta \\
&= \int \sec^3\theta\, d\theta \\
&= \frac{1}{2}\tan\theta\sec\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|
\end{aligned}$$

where the integral of $\sec^3\theta$ is determined via a reduction formula. Using the fact that $u = \tan\theta$ we find that $\sec\theta = \sqrt{1 + u^2}$ using a Pythagorean identity. Therefore,

$$\int \sqrt{1 + u^2}\, du = \frac{1}{2}u\sqrt{1 + u^2} + \ln\left|\sqrt{1 + u^2} + u\right|$$

The surface area is then:

$$\begin{aligned}
\text{Surface Area} &= 4\pi \int_0^1 \sqrt{1+u^2}\, du \\
&= 4\pi \left[\frac{1}{2}u\sqrt{1+u^2} + \ln\left|\sqrt{1+u^2}+u\right|\right]_0^1 \\
&= 4\pi \left[\frac{1}{2}(1)\sqrt{1+1^2} + \ln\left|\sqrt{1+1^2}+1\right|\right] - 4\pi \left[\frac{1}{2}(0)\sqrt{1+0^2} + \ln\left|\sqrt{1+0^2}+0\right|\right] \\
&= 4\pi \left[\frac{\sqrt{2}}{2} + \ln 2\right] - 4\pi \Big[0 + \ln 1\Big] \\
&= \boxed{2\pi\sqrt{2} + 2\pi \ln 2}
\end{aligned}$$

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Problem 6 Solution

6.

(a) Estimate $\ln\frac{3}{2}$ using the degree two Taylor polynomial for $y = \ln x$ around $x = 1$.

(b) Estimate $\ln\frac{3}{2}$ using the Midpoint rule with $n = 2$ for the integral $\int_1^{3/2}\frac{dx}{x}$.

(c) Calculate the error bounds for the two estimates. Does this tell you which is closer to the exact answer?

Solution:

(a) The degree two Taylor polynomial for $f(x) = \ln x$ around $x = 1$ has the formula:

$$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$$

The derivatives of $f(x)$ evaluated at $x = 1$ are:

k	$f^{(k)}(x)$	$f^{(k)}(1)$
0	$\ln x$	$\ln 1 = 0$
1	$\frac{1}{x}$	$\frac{1}{1} = 1$
2	$-\frac{1}{x^2}$	$-\frac{1}{1^2} = -1$

The Taylor polynomial $T_2(x)$ is then:

$$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$$
$$T_2(x) = 0 + (x-1) - \frac{1}{2!}(x-1)^2$$
$$T_2(x) = (x-1) - \frac{1}{2}(x-1)^2$$

We will now estimate $\ln\frac{3}{2}$ using $T_2(\frac{3}{2})$.

$$\ln\frac{3}{2} \approx T_2\left(\frac{3}{2}\right) \approx \left(\frac{3}{2}-1\right) - \frac{1}{2}\left(\frac{3}{2}-1\right)^2 \approx \boxed{\frac{3}{8}}$$

(b) The value of Δx in the Midpoint rule is:

$$\Delta x = \frac{b-a}{n} = \frac{\frac{3}{2}-1}{2} = \frac{1}{4}$$

The Midpoint estimate M_2 is:

$$\begin{aligned} M_2 &= \Delta x \left[f\left(\frac{9}{8}\right) + f\left(\frac{11}{8}\right) \right] \\ &= \frac{1}{4}\left[\frac{1}{\frac{9}{8}} + \frac{1}{\frac{11}{8}} \right] \\ &= \frac{1}{4}\left[\frac{8}{9} + \frac{8}{11} \right] \\ &= \boxed{\frac{40}{99}} \end{aligned}$$

(c) The error bound for part (a) is given by the formula:

$$\text{Error} \le K\frac{|x-a|^{n+1}}{(n+1)!}$$

where $x = \frac{3}{2}$, $a = 1$, $n = 2$, and K satisfies the inequality $|f'''(u)| \le K$ for all $u \in [1, \frac{3}{2}]$. One can show that $f'''(x) = \frac{2}{x^3}$. We conclude that $|f'''(u)| = |\frac{2}{u^3}| < 2$ for all $u \in [1, \frac{3}{2}]$. So we choose $K = 2$ and the error bound is:

$$\text{Error} \le 2\frac{|\frac{3}{2}-1|^3}{3!} = \frac{1}{24}$$

The error bound for part (b) is given by the formula:

$$\text{Error}(M_n) \le \frac{K(b-a)^3}{24n^2}$$

where $a = 1$, $b = \frac{3}{2}$, $n = 2$, and K satisfies the inequality $|(\frac{1}{x})''| \le K$ for all $x \in [1, \frac{3}{2}]$. We conclude that $|(\frac{1}{x})''| = |\frac{2}{x^3}| \le 2$ for all $x \in [1, \frac{3}{2}]$. So we choose $K = 2$ and the error bound is:

$$\text{Error}(M_2) \le \frac{2 \cdot (\frac{3}{2}-1)^3}{24(2)^2} = \frac{1}{384}$$

We cannot tell which of $\frac{3}{8}$ and $\frac{40}{99}$ is closer to the exact answer. All we know is that both errors are smaller than $\frac{1}{24}$.

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Problem 7 Solution

7. Does the improper integral $\displaystyle\int_0^{+\infty} \frac{dx}{1+x^3}$ converge or diverge? Justify your answer.

Solution: We begin by rewriting the integral as follows:

$$\int_0^{+\infty} \frac{dx}{1+x^3} = \int_0^{1} \frac{dx}{1+x^3} + \int_1^{+\infty} \frac{dx}{1+x^3}$$

The first integral on the right hand side is a proper integral so we know that it converges. We will use the Comparison Test to show that the second integral converges. Let $g(x) = \frac{1}{1+x^3}$. We must choose a function $f(x)$ that satisfies:

$$(1) \int_1^{+\infty} f(x)\,dx \text{ converges} \quad \text{and} \quad (2)\ \ 0 \le g(x) \le f(x) \text{ for } x \ge 1$$

We choose $f(x) = \frac{1}{x^3}$. This function satisfies the inequality:

$$\begin{aligned} 0 &\le g(x) \le f(x) \\ 0 &\le \frac{1}{1+x^3} \le \frac{1}{x^3} \end{aligned}$$

for $x \ge 1$ because the denominator of $g(x)$ is greater than the denominator of $f(x)$ for these values of x. Furthermore, the integral $\int_1^{+\infty} f(x)\,dx = \int_1^{+\infty} \frac{1}{x^3}\,dx$ converges because it is a p-integral with $p = 3 > 1$. Therefore, the integral $\int_1^{+\infty} g(x)\,dx = \int_1^{+\infty} \frac{1}{1+x^3}\,dx$ converges by the Comparison Test and the integral $\int_0^{+\infty} \frac{dx}{1+x^3}$ **converges**.

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Problem 8 Solution

8. What is the arc-length of the segment of the parabola $y = 4 - x^2$ above the x-axis?

Solution: The arclength is:

$$\begin{aligned} L &= \int_a^b \sqrt{1 + f'(x)^2}\, dx \\ &= \int_{-2}^{2} \sqrt{1 + (-2x)^2}\, dx \\ &= \int_{-2}^{2} \sqrt{1 + 4x^2}\, dx \\ &= 2\int_0^2 \sqrt{1 + 4x^2}\, dx \end{aligned}$$

We solve the integral using the trigonometric substitution $x = \frac{1}{2}\tan\theta$, $dx = \frac{1}{2}\sec^2\theta\, d\theta$. The indefinite integral is then:

$$\begin{aligned} \int \sqrt{1 + 4x^2}\, dx &= \int \sqrt{1 + 4\left(\frac{1}{2}\tan\theta\right)^2} \left(\frac{1}{2}\sec^2\theta\, d\theta\right) \\ &= \frac{1}{2}\int \sqrt{1 + \tan^2\theta}\ \sec^2\theta\, d\theta \\ &= \frac{1}{2}\int \sqrt{\sec^2\theta}\ \sec^2\theta\, d\theta \\ &= \frac{1}{2}\int \sec^3\theta\, d\theta \\ &= \frac{1}{4}\tan\theta\sec\theta + \frac{1}{4}\ln|\sec\theta + \tan\theta| \end{aligned}$$

Using the fact that $x = \frac{1}{2}\tan\theta$ we find that $\tan\theta = 2x$ and $\sec\theta = \sqrt{1 + 4x^2}$ either using a triangle or a Pythagorean identity. The integral in terms of x is then:

$$\int \sqrt{1 + 4x^2}\, dx = \frac{1}{4}\tan\theta\sec\theta + \frac{1}{4}\ln|\sec\theta + \tan\theta| = \frac{1}{2}x\sqrt{1 + 4x^2} + \frac{1}{4}\ln\left|\sqrt{1 + 4x^2} + 2x\right|$$

The arclength is then:

$$
\begin{aligned}
L &= 2\int_0^2 \sqrt{1+4x^2}\,dx \\
&= 2\left[\frac{1}{2}x\sqrt{1+4x^2} + \frac{1}{4}\ln\left|\sqrt{1+4x^2}+2x\right|\right]_0^2 \\
&= \left[x\sqrt{1+4x^2} + \frac{1}{2}\ln\left|\sqrt{1+4x^2}+2x\right|\right]_0^2 \\
&= \left[2\sqrt{1+4(2)^2} + \frac{1}{2}\ln\left|\sqrt{1+4(2)^2}+2(2)\right|\right] - \left[0\cdot\sqrt{1+4(0)^2} + \frac{1}{2}\ln\left|\sqrt{1+4(0)^2}+2(0)\right|\right] \\
&= \boxed{2\sqrt{17} + \frac{1}{2}\ln\left(\sqrt{17}+4\right)}
\end{aligned}
$$

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Problem 9 Solution

9. Find a formula for the general Taylor polynomial $T_n(x)$ for the following functions around the specified points:

(a) e^{-x^2} around $x = 0$

(b) $\sqrt{x}$ around $x = 1$

Solution:

(a) We'll use a shortcut to find $T_n(x)$. We'll start with the general Maclaurin polynomial for e^x which is:

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}$$

and replace x with $-x^2$ to get:

$$T_n(x) = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + \frac{(-x^2)^n}{n!} = \boxed{\sum_{k=0}^{n} \frac{(-x^2)^k}{k!}}$$

(b) The function $f(x)$ and its derivatives evaluated at $a = 1$ are:

k	$f^{(k)}(x)$	$f^{(k)}(1)$
0	$x^{1/2}$	1
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{2}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{2^2}$
3	$\frac{3}{8}x^{-5/2}$	$\frac{1 \cdot 3}{2^3}$
4	$-\frac{15}{16}x^{-7/2}$	$-\frac{1 \cdot 3 \cdot 5}{2^4}$

The Taylor polynomial of degree n is:

$$T_n(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{2^2 2!}(x-1)^2 + \frac{1 \cdot 3}{2^3 3!}(x-1)^3 - \frac{1 \cdot 3 \cdot 5}{2^4 4!} + \ldots$$

$$= \boxed{1 + \frac{1}{2}(x-1) + \sum_{k=2}^{n} (-1)^{k-1} \frac{1 \cdot 3 \cdot \ldots \cdot (2k-3)}{2^k k!}(x-1)^k}$$