**Math 181, Exam 2, Study Guide 3**
**Problem 1 Solution**

1. Compute the improper integral: $\displaystyle\int_0^{+\infty} xe^{-x}\,dx$.

**Solution**: We evaluate the integral by turning it into a limit calculation.

$$\int_0^{+\infty} xe^{-x}\,dx = \lim_{R\to+\infty}\int_0^R xe^{-x}\,dx$$

We use Integration by Parts to compute the integral. Let $u = x$ and $v' = e^{-x}$. Then $u' = 1$ and $v = -e^{-x}$. Using the Integration by Parts formula we get:

$$\begin{aligned}
\int_a^b uv'\,dx &= \Big[uv\Big]_a^b - \int_a^b u'v\,dx \\
\int_0^R xe^{-x}\,dx &= \Big[-xe^{-x}\Big]_0^R - \int_0^R \left(-e^{-x}\right)\,dx \\
&= \Big[-xe^{-x}\Big]_0^R + \int_0^R e^{-x}\,dx \\
&= \Big[-xe^{-x}\Big]_0^R + \Big[-e^{-x}\Big]_0^R \\
&= \Big[-Re^{-R} + 0e^{-0}\Big] + \Big[-e^{-R} + e^{-0}\Big] \\
&= -\frac{R}{e^R} - \frac{1}{e^R} + 1
\end{aligned}$$

We now take the limit of the above function as $R \to +\infty$.

$$\begin{aligned}
\int_0^{+\infty} xe^{-x}\,dx &= \lim_{R\to+\infty}\int_0^R xe^{-x}\,dx \\
&= \lim_{R\to+\infty}\left(-\frac{R}{e^R} - \frac{1}{e^R} + 1\right) \\
&= -\lim_{R\to+\infty}\frac{R}{e^R} - \lim_{R\to+\infty}\frac{1}{e^R} + 1 \\
&= -\lim_{R\to+\infty}\frac{R}{e^R} - 0 + 1 \\
&\overset{\text{L'H}}{=} -\lim_{R\to+\infty}\frac{(R)'}{(e^R)'} - 0 + 1 \\
&= -\lim_{R\to+\infty}\frac{1}{e^R} - 0 + 1 \\
&= -0 - 0 + 1 \\
&= \boxed{1}
\end{aligned}$$

**Math 181, Exam 2, Study Guide 3**
**Problem 2 Solution**

2. Compute the improper integral: $\displaystyle\int_0^{+\infty} \frac{dx}{x^2+1}$.

**Solution**: We evaluate the integral by turning it into a limit calculation.

$$\int_0^{+\infty} \frac{dx}{x^2+1} = \lim_{R\to+\infty} \int_0^R \frac{dx}{x^2+1}$$

The integral has a simple antiderivative so its value is:

$$\begin{aligned}\int_0^R \frac{dx}{x^2+1} &= \Big[\arctan x\Big]_0^R \\ &= \arctan R - \arctan 0 \\ &= \arctan R\end{aligned}$$

We now take the limit of the above function as $R \to +\infty$.

$$\begin{aligned}\int_0^{+\infty} \frac{dx}{x^2+1} &= \lim_{R\to+\infty} \int_0^R \frac{dx}{x^2+1} \\ &= \lim_{R\to+\infty} \arctan R \\ &= \boxed{\frac{\pi}{2}}\end{aligned}$$

**Math 181, Exam 2, Study Guide 3**
**Problem 3 Solution**

3. Determine whether the improper integral $\displaystyle\int_0^{+\infty} \frac{x^2+2x+5}{x^3+x+1}$ converges or not.

**Solution**: We will show that the integral diverges using the Comparison Test. First, we rewrite the integral as:

$$\int_0^{+\infty} \frac{x^2+2x+5}{x^3+x+1}\,dx = \int_0^{1} \frac{x^2+2x+5}{x^3+x+1}\,dx + \int_1^{+\infty} \frac{x^2+2x+5}{x^3+x+1}\,dx$$

The first integral on the right hand side is a proper integral so it converges. We will focus on the second integral now. We guess that it diverges so we let $f(x) = \frac{x^2+2x+5}{x^3+x+1}$. We must choose a function $g(x)$ such that:

$$(1)\quad \int_1^{+\infty} g(x)\,dx \text{ diverges} \qquad \text{and} \qquad (2)\quad 0 \le g(x) \le f(x) \text{ for } x \ge 1$$

We choose the function $g(x)$ by using the fact that $0 \le x^2 \le x^2+2x+5$ and $0 \le x^3+x+1 \le x^3+x^3+x^3$ for $x \ge 1$. Then we get:

$$\begin{aligned}
0 &\le \frac{x^2}{x^3+x^3+x^3} \le \frac{x^2+2x+5}{x^3+x+1}\\
0 &\le \frac{x^2}{3x^3} \le \frac{x^2+2x+5}{x^3+x+1}\\
0 &\le \frac{1}{3x} \le \frac{x^2+2x+5}{x^3+x+1}
\end{aligned}$$

So we choose $g(x) = \frac{1}{3x}$ so that $0 \le g(x) \le f(x)$ for $x \ge 1$. Furthermore, we know that:

$$\int_1^{+\infty} g(x)\,dx = \int_1^{+\infty} \frac{1}{3x}\,dx = \frac{1}{3}\int_1^{+\infty} \frac{1}{x}\,dx$$

diverges because this is a $p$-integral with $p = 1 \le 1$. Thus, the integral $\int_1^{+\infty} \frac{x^2+2x+5}{x^3+x+1}\,dx$ diverges by the Comparison Test. Since this integral diverges we know that $\int_0^{+\infty} \frac{x^2+2x+5}{x^3+x+1}\,dx$ also **diverges**.

**Math 181, Exam 2, Study Guide 3**
**Problem 4 Solution**

4. Determine whether the improper integral $\displaystyle\int_0^1 \frac{e^x}{\sqrt{x}}\,dx$ converges or not.

**Solution**: We will show that the integral converges using the Comparison Test. Let $g(x) = \frac{e^x}{\sqrt{x}}$. We must choose a function $f(x)$ such that:

$$(1)\ \int_0^1 f(x)\,dx \text{ converges} \quad \text{and} \quad (2)\ \ 0 \le g(x) \le f(x) \text{ for } 0 < x \le 1$$

We choose $f(x) = \frac{e}{\sqrt{x}}$. This choice of $f(x)$ satisfies the inequality:

$$\begin{aligned} 0 &\le g(x) \le f(x) \\ 0 &\le \frac{e^x}{\sqrt{x}} \le \frac{e}{\sqrt{x}} \end{aligned}$$

for $0 < x \le 1$ because $e^x \le e$ for these values of $x$. Furthermore, the integral $\int_0^1 f(x)\,dx = \int_0^1 \frac{e}{\sqrt{x}}\,dx = e\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges because it is a $p$-integral with $p = \frac{1}{2} < 1$. Therefore, the integral $\int_0^1 g(x)\,dx = \int_0^1 \frac{e^x}{\sqrt{x}}$ **converges** by the Comparison Test.

**Math 181, Exam 2, Study Guide 3**
**Problem 5 Solution**

5. Determine whether the improper integral $\displaystyle\int_1^{+\infty} \frac{dx}{\sqrt{x^3+x+1}}$ converges or not.

**Solution**: We will show that the integral converges using the Comparison Test. Let $g(x) = \frac{1}{\sqrt{x^3+x+1}}$. We must choose a function $f(x)$ such that:

$$(1) \int_1^{+\infty} f(x)\,dx \text{ converges} \quad \text{and} \quad (2)\ \ 0 \le g(x) \le f(x) \text{ for } x \ge 1$$

We choose $f(x) = \frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}}$. This choice of $f(x)$ satisfies the inequality:

$$\begin{aligned} &0 \le g(x) \le f(x) \\ &0 \le \frac{1}{\sqrt{x^3+x+1}} \le \frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}} \end{aligned}$$

for $x \ge 1$ using the argument that the denominator of $g(x)$ is greater than the denominator of $f(x)$ for these values of $x$. Furthermore, the integral $\int_1^{+\infty} f(x)\,dx = \int_1^{+\infty} \frac{1}{x^{3/2}}\,dx$ converges because it is a $p$-integral with $p = \frac{3}{2} > 1$. Therefore, the integral $\int_1^{+\infty} g(x)\,dx = \int_1^{+\infty} \frac{1}{\sqrt{x^3+x+1}}\,dx$ **converges** by the Comparison Test.

**Math 181, Exam 2, Study Guide 3**
**Problem 6 Solution**

6. Compute the arclength of the graph of the function $f(x) = x^{3/2}$ from $x = 0$ to $x = 1$.

**Solution**: The arclength is:

$$\begin{aligned} L &= \int_a^b \sqrt{1 + f'(x)^2}\, dx \\ &= \int_0^1 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\, dx \\ &= \int_0^1 \sqrt{1 + \frac{9}{4}x}\, dx \end{aligned}$$

We now use the $u$-substitution $u = 1 + \frac{9}{4}x$. Then $\frac{4}{9}\, du = dx$, the lower limit of integration changes from 0 to 1, and the upper limit of integration changes from 1 to $\frac{13}{4}$.

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \frac{9}{4}x}\, dx \\ &= \frac{4}{9}\int_1^{13/4} \sqrt{u}\, du \\ &= \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{13/4} \\ &= \frac{4}{9}\left[\frac{2}{3}\left(\frac{13}{4}\right)^{3/2} - \frac{2}{3}(1)^{3/2}\right] \\ &= \boxed{\frac{8}{27}\left[\left(\frac{13}{4}\right)^{3/2} - 1\right]} \end{aligned}$$

**Math 181, Exam 2, Study Guide 3**
**Problem 7 Solution**

7. Compute the arclength of the graph of the function $f(x) = x^{1/2} - \frac{1}{3}x^{3/2}$ on the interval $[1, 4]$.

**Solution**: The arclength is:

$$
\begin{aligned}
L &= \int_a^b \sqrt{1 + f'(x)^2}\, dx \\
&= \int_1^4 \sqrt{1 + \left(\frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2}\right)^2}\, dx \\
&= \int_1^4 \sqrt{1 + \frac{1}{4}x^{-1} - \frac{1}{2} + \frac{1}{4}x}\, dx \\
&= \int_1^4 \sqrt{\frac{1}{4}x^{-1} + \frac{1}{2} + \frac{1}{4}x}\, dx \\
&= \int_1^4 \sqrt{\left(\frac{1}{2}x^{-1/2} + \frac{1}{2}x^{1/2}\right)^2}\, dx \\
&= \int_1^4 \left(\frac{1}{2}x^{-1/2} + \frac{1}{2}x^{1/2}\right) dx \\
&= \frac{1}{2}\int_1^4 \left(x^{-1/2} + x^{1/2}\right) dx \\
&= \frac{1}{2}\left[2\sqrt{x} + \frac{2}{3}x^{3/2}\right]_1^4 \\
&= \frac{1}{2}\left[\left(2\sqrt{4} + \frac{2}{3}(4)^{3/2}\right) - \left(2\sqrt{1} + \frac{2}{3}(1)^{3/2}\right)\right] \\
&= \frac{1}{2}\left[\left(4 + \frac{16}{3}\right) - \left(2 + \frac{2}{3}\right)\right] \\
&= \boxed{\frac{10}{3}}
\end{aligned}
$$

**Math 181, Exam 2, Study Guide 3**
**Problem 8 Solution**

8. Compute the surface area of the surface obtained by rotating the curve $y = x^3$ on $[0, 1]$ about the $x$-axis.

**Solution**: The surface area formula is:

$$\text{Surface Area} = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\,dx$$

Using $a = 0$, $b = 1$, $f(x) = x^3$, and $f'(x) = 3x^2$ we get:

$$\begin{aligned}\text{Surface Area} &= 2\pi \int_0^1 x^3\sqrt{1 + (3x^2)^2}\,dx \\ &= 2\pi \int_0^1 x^3\sqrt{1 + 9x^4}\,dx\end{aligned}$$

To evaluate the integral we use the $u$-substitution $u = 1 + 9x^4$. Then $\frac{1}{36}\,du = x^3\,dx$, the lower limit of integration changes from 0 to 1, and the upper limit changes from 1 to 10. Making these substitutions we get:

$$\begin{aligned}\text{Surface Area} &= \frac{2\pi}{36}\int_1^{10} \sqrt{u}\,du \\ &= \frac{\pi}{18}\left[\frac{2}{3}u^{3/2}\right]_1^{10} \\ &= \frac{\pi}{18}\left[\frac{2}{3}(10)^{3/2} - \frac{2}{3}(1)^{3/2}\right] \\ &= \boxed{\frac{\pi}{27}\left[(10)^{3/2} - 1\right]}\end{aligned}$$

**Math 181, Exam 2, Study Guide 3**
**Problem 9 Solution**

9. Find the center of mass of the constant density lamina lying below the semicircle $y^2+x^2 = 1$ with $y \geq 0$ and above the $x$-axis.

**Solution**: The coordinates of the center of mass are given by the formulas:

$$x_{CM} = \frac{M_y}{M}, \qquad y_{CM} = \frac{M_x}{M}$$

Using symmetry, we know that $M_y = 0$ so we get $x_{CM} = 0$. The mass of the lamina is:

$$M = \rho A = \rho \cdot \frac{1}{2}\pi(1)^2 = \frac{1}{2}\rho\pi$$

The value of $M_x$ is:

$$\begin{aligned}
M_x &= \frac{1}{2}\rho \int_a^b f(x)^2 \, dx \\
&= \frac{1}{2}\rho \int_{-1}^{1} \left(\sqrt{1-x^2}\right)^2 \, dx \\
&= \frac{1}{2}\rho \int_{-1}^{1} \left(1-x^2\right) \, dx \\
&= \frac{1}{2}\rho \left[x - \frac{1}{3}x^3\right]_{-1}^{1} \\
&= \frac{1}{2}\rho \left[\left(1 - \frac{1}{3}(1)^3\right) - \left(-1 - \frac{1}{3}(-1)^3\right)\right] \\
&= \frac{1}{2}\rho \left[\frac{2}{3} - \left(-\frac{2}{3}\right)\right] \\
&= \frac{2}{3}\rho
\end{aligned}$$

The center of mass coordinates are then:

$$\begin{aligned}
x_{CM} &= \frac{M_y}{M} = \boxed{0} \\
y_{CM} &= \frac{M_x}{M} = \frac{\frac{2}{3}\rho}{\frac{1}{2}\rho\pi} = \boxed{\frac{4}{3\pi}}
\end{aligned}$$

**Math 181, Exam 2, Study Guide 3**
**Problem 10 Solution**

10. Compute the 2nd Taylor polynomial of the function $x^3 - 3x^2 + 5x + 3$ centered at the point $a = 1$.

**Solution**: The 2nd degree Taylor polynomial $T_2(x)$ of $f(x)$ centered at $a = 1$ has the formula:

$$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$$

The derivatives of $f(x)$ and their values at $x = 1$ are:

| $k$ | $f^{(k)}(x)$ | $f^{(k)}(1)$ |
|---|---|---|
| 0 | $x^3 - 3x^2 + 5x + 3$ | $1^3 - 3(1)^2 + 5(1) - 2 = 6$ |
| 1 | $3x^2 - 6x + 5$ | $3(1)^2 - 6(1) + 5 = 2$ |
| 2 | $6x - 6$ | $6(1) - 6 = 0$ |

The function $T_2(x)$ is then:

$$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$$
$$T_2(x) = 6 + 2(x-1) + \frac{0}{2!}(x-1)^2$$
$$\boxed{T_2(x) = 6 + 2(x-1)}$$

**Math 181, Exam 2, Study Guide 3**
**Problem 11 Solution**

11. Compute the 3rd Taylor polynomial of the function $f(x) = \ln x$ centered at $a = 1$.

**Solution**: The 3rd degree Maclaurin polynomial $T_3(x)$ of $f(x)$ has the formula:

$$T_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

The derivatives of $f(x)$ and their values at $x = 1$ are:

| $k$ | $f^{(k)}(x)$ | $f^{(k)}(1)$ |
|---|---|---|
| 0 | $\ln x$ | $\ln 1 = 0$ |
| 1 | $\frac{1}{x}$ | $\frac{1}{1} = 1$ |
| 2 | $-\frac{1}{x^2}$ | $-\frac{1}{1^2} = -1$ |
| 3 | $\frac{2}{x^3}$ | $\frac{2}{1^3} = 2$ |

The function $T_3(x)$ is then:

$$T_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$
$$T_3(x) = 0 + (x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$$
$$\boxed{T_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3}$$

**Math 181, Exam 2, Study Guide 3**
**Problem 12 Solution**

12. Compute the 6th Maclaurin polynomial of the function $f(x) = \cos x$.

**Solution**: The 6th degree Maclaurin polynomial $T_3(x)$ of $f(x)$ has the formula:

$$T_6(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \frac{f^{(6)}(0)}{6!}x^6$$

The derivatives of $f(x)$ and their values at $x = 0$ are:

| $k$ | $f^{(k)}(x)$ | $f^{(k)}(0)$ |
|---|---|---|
| 0 | $\cos x$ | $\cos 0 = 1$ |
| 1 | $-\sin x$ | $-\sin 0 = 0$ |
| 2 | $-\cos x$ | $-\cos 0 = -1$ |
| 3 | $\sin x$ | $\sin 0 = 0$ |
| 4 | $\cos x$ | $\cos 0 = 1$ |
| 5 | $-\sin x$ | $-\sin 0 = 0$ |
| 6 | $-\cos x$ | $-\cos 0 = -1$ |

The function $T_3(x)$ is then:

$$T_6(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \frac{f^{(6)}(0)}{6!}x^6$$
$$T_6(x) = 1 + 0 \cdot x - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \frac{0}{5!}x^5 - \frac{1}{6!}x^6$$
$$\boxed{T_6(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6}$$

**Math 181, Exam 2, Study Guide 3**
**Problem 13 Solution**

13. Give an upper bound for the error $|\cos 1 - T_4(1)|$ where $T_4(x)$ is the 4th Maclaurin polynomial of the function $\cos x$.

**Solution**: The error bound is given by the formula:

$$\text{Error} \leq K\frac{|x-a|^{n+1}}{(n+1)!}$$

where $x = 1$, $a = 0$, $n = 4$, and $K$ satisfies the inequality $|f^{(5)}(u)| \leq K$ for all $u \in [0,1]$. One can show that $f^{(5)}(x) = -\sin x$. We conclude that $|f'''(u)| = |-\sin u| \leq 1$ for all $u \in [0,1]$. So we choose $K = 1$ and the error bound is:

$$\text{Error} \leq 1 \cdot \frac{|1-0|^5}{5!} = \boxed{\frac{1}{120}}$$

**Math 181, Exam 2, Study Guide 3**
**Problem 14 Solution**

14. Compute the sum of the series $\dfrac{1}{3} - \dfrac{2}{9} + \dfrac{4}{27} - \dfrac{8}{81} + \cdots$.

**Solution**: We recognize the given series as a geometric series. In order to find its sum we must first rewrite the series.

$$\begin{aligned} \frac{1}{3} - \frac{2}{9} + \frac{4}{27} - \frac{8}{81} + \cdots &= \frac{2^0}{3^1} - \frac{2^1}{3^2} + \frac{2^2}{3^3} - \frac{2^3}{3^4} + \cdots \\ &= \sum_{n=0}^{+\infty} \frac{(-1)^n 2^n}{3^{n+1}} \\ &= \sum_{n=0} \frac{1}{3}\left(-\frac{2}{3}\right)^n \end{aligned}$$

The series converges because $|r| = |-\frac{2}{3}| < 1$. We can now use the formula:

$$\sum_{n=0}^{+\infty} cr^n = \frac{c}{1-r}$$

where $c = \frac{1}{3}$ and $r = -\frac{2}{3}$. The sum of the series is:

$$\sum_{n=0}^{+\infty} \frac{1}{3}\left(-\frac{2}{3}\right)^n = \frac{\frac{1}{3}}{1-(-\frac{2}{3})} = \boxed{\frac{1}{5}}$$

**Math 181, Exam 2, Study Guide 3**
**Problem 15 Solution**

15. Compute the sum of the series: $\displaystyle\sum_{n=1}^{+\infty} \frac{2}{n(n+2)}$.

**Solution**: We recognize the given series as a telescoping series. Our goal is to construct a formula for the $N$th partial sum, $S_N$, and find its limit as $N \to \infty$.

We begin by decomposing $f(n)$ into a sum of simpler fractions.

$$\begin{aligned}\frac{2}{n(n+2)} &= \frac{A}{n} + \frac{B}{n+2}\\ 2 &= A(n+2) + Bn\end{aligned}$$

We now plug in $n = 0$ and $n = -2$ to find the values of $A$ and $B$.

$$\begin{aligned} n = 0:&\ \ 2 = A(0+2) + B(0) \quad \Rightarrow \quad A = 1\\ n = -2:&\ \ 2 = A(-2+2) + B(-2) \quad \Rightarrow \quad B = -1\end{aligned}$$

Thus, we can rewrite the series as:

$$\sum_{n=1}^{+\infty} \frac{2}{n(n+2)} = \sum_{n=1}^{+\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$$

A formula for $S_N$ is:

$$\begin{aligned} S_N &= \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+2}\right)\\ S_N &= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots\\ &\quad + \left(\frac{1}{N-3} - \frac{1}{N-1}\right) + \left(\frac{1}{N-2} - \frac{1}{N}\right) + \left(\frac{1}{N-1} - \frac{1}{N+1}\right) + \left(\frac{1}{N} - \frac{1}{N+2}\right)\\ S_N &= 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}\end{aligned}$$

Taking the limit as $N \to \infty$ we get:

$$\begin{aligned}\sum_{n=1}^{+\infty} \frac{2}{n(n+2)} &= \sum_{n=1}^{+\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)\\ &= \lim_{N\to\infty} S_N\\ &= \lim_{N\to\infty} \left(1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}\right)\\ &= 1 + \frac{1}{2} - 0 - 0\\ &= \boxed{\frac{3}{2}}\end{aligned}$$

**Math 181, Exam 2, Study Guide 3**
**Problem 16 Solution**

16. Determine whether the series $\displaystyle\sum_{n=1}^{+\infty} \frac{n^2+1}{n^3+n+1}$ converges or not.

**Solution**: We use the Comparison Test to determine whether the first series converges or not. We guess that the series diverges. Let $b_n = \frac{n^2+1}{n^3+n+1}$. We must choose a series $\sum a_n$ such that (1) $0 \le a_n \le b_n$ for $n \ge 1$ and (2) $\sum_{n=1}^{+\infty} a_n$ diverges. We notice that:

$$0 \le \frac{n^2}{n^3+n^3+n^3} = \frac{1}{3n} \le \frac{n^2+1}{n^3+n+1}$$

for $n \ge 1$. So we choose $a_n = \frac{1}{3n}$. Since the series $\sum_{n=1}^{+\infty} \frac{1}{3n} = \frac{1}{3}\sum_{n=1}^{+\infty} \frac{1}{n}$ diverges because it is a $p$-series with $p = 1 \le 1$, the series $\sum_{n=1}^{+\infty} \frac{n^2+1}{n^3+n+1}$ **diverges** by the Comparison Test.

**Math 181, Exam 2, Study Guide 3**
**Problem 17 Solution**

17. Determine whether the series $\displaystyle\sum_{n=1}^{+\infty} \frac{\ln n}{n^2}$ converges or not.

**Solution**: We use the Comparison Test to determine whether the first series converges or not. We guess that the series converges. Let $a_n = \frac{\ln n}{n^2}$. We must choose a series $\sum b_n$ such that (1) $0 \le a_n \le b_n$ for $n \ge 1$ and (2) $\sum_{n=1}^{+\infty} b_n$ converges. We notice that:

$$0 \le \frac{\ln n}{n^2} \le \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$$

for $n \ge 1$. So we choose $b_n = \frac{1}{n^{3/2}}$. Since the series $\sum_{n=1}^{+\infty} \frac{1}{n^{3/2}}$ converges because it is a $p$-series with $p = \frac{3}{2} > 1$, the series $\sum_{n=1}^{+\infty} \frac{\ln n}{n^2}$ **converges** by the Comparison Test.

**Math 181, Exam 2, Study Guide 3**
**Problem 18 Solution**

18. Determine whether the series $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{n^2 + \cos(\pi n)}$ converges or not.

**Solution**: We use the Comparison Test to determine whether the first series converges or not. We guess that the series converges. Let $a_n = \frac{1}{n^2+\cos(\pi n)}$. We must choose a series $\sum b_n$ such that (1) $0 \le a_n \le b_n$ for $n \ge 2$ and (2) $\sum_{n=2}^{+\infty} b_n$ converges. We notice that:

$$0 \le \frac{1}{n^2 + \cos(\pi n)} \le \frac{1}{n^{3/2}}$$

for $n \ge 2$. So we choose $b_n = \frac{1}{n^{3/2}}$. Since the series $\sum_{n=2}^{+\infty} \frac{1}{n^{3/2}}$ converges because it is a $p$-series with $p = \frac{3}{2} > 1$, the series $\sum_{n=2}^{+\infty} \frac{\ln n}{n^2}$ **converges** by the Comparison Test.

**Math 181, Exam 2, Study Guide 3**
**Problem 19 Solution**

19. Determine whether the series $\displaystyle\sum_{n=2}^{+\infty} \frac{\sin n}{n^2}$ converges or not.

**Solution**: The series does not consist of all positive terms so we first check for absolute convergence by considering the series of absolute values:

$$\sum_{n=2}^{+\infty} \left| \frac{\sin n}{n^2} \right|$$

We notice that:

$$0 \leq \left| \frac{\sin n}{n^2} \right| \leq \frac{1}{n^2}$$

for $n \geq 2$. Furthermore, $\sum_{n=2}^{+\infty} \frac{1}{n^2}$ converges because it is a $p$-series with $p = 2 > 1$. Therefore, the series $\sum_{n=2}^{+\infty} \frac{\sin n}{n^2}$ is absolutely convergent which means that it **converges**.

**Math 181, Exam 2, Study Guide 3**
**Problem 20 Solution**

20. Determine whether the series $\sum_{n=1}^{+\infty} \frac{\cos(\pi n)}{\sqrt{n}}$ converges or not.

**Solution**: The series can be rewritten as:

$$\sum_{n=1}^{+\infty} \frac{\cos(\pi n)}{\sqrt{n}} = \sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt{n}}$$

The series alternating so we check for convergence using the Leibniz Test. Let $a_n = \frac{1}{\sqrt{n}}$. We know that $a_n$ is decreasing for $n \geq 1$ and that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

Therefore, the series $\sum_{n=1}^{+\infty} \frac{\cos(\pi n)}{\sqrt{n}}$ **converges** by the Leibniz Test.

**Math 181, Exam 2, Study Guide 3**
**Problem 21 Solution**

21. Determine whether the series $\displaystyle\sum_{n=1}^{+\infty} \frac{(n!)^2}{(2n)!}$ converges or not.

**Solution**: We use the Ratio Test to determine whether or not the series converges.

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n\to\infty} \frac{((n+1)!)^2}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2} \\
&= \lim_{n\to\infty} \frac{(n+1)!(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!n!} \\
&= \lim_{n\to\infty} \frac{(n+1)n!(n+1)n!}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{n!n!} \\
&= \lim_{n\to\infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)} \\
&= \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} \\
&= \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{4+\frac{3}{2n}+\frac{1}{2n^2}} \\
&= \frac{1}{4}
\end{aligned}$$

Since $\rho = \frac{1}{4} < 1$, the series **converges** by the Ratio Test.

**Math 181, Exam 2, Study Guide 3**
**Problem 22 Solution**

22. Determine whether the series $\displaystyle\sum_{n=1}^{+\infty} \frac{3^n}{n^4}$ converges or not.

**Solution**: We use the Ratio Test to determine whether or not the series converges.

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^4} \cdot \frac{n^4}{3^n} \\
&= \lim_{n\to\infty} \frac{3^n 3^1}{3^n} \cdot \frac{(n+1)^4}{n^4} \\
&= \lim_{n\to\infty} 3\left(\frac{n+1}{n}\right)^4 \\
&= 3 \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^4 \\
&= 3
\end{aligned}$$

Since $\rho = 3 > 1$, the series **diverges** by the Ratio Test.

**Math 181, Exam 2, Study Guide 3**
**Problem 23 Solution**

23. Determine whether the series $\displaystyle\sum_{n=2}^{+\infty} \frac{(-1)^n}{n \ln n}$

**Solution**: The series is alternating so we first check for absolute convergence by considering the series of absolute values:

$$\sum_{n=2}^{+\infty} \left| \frac{(-1)^n}{n \ln n} \right| = \sum_{n=2}^{+\infty} \frac{1}{n \ln n}$$

We use the Integral Test to determine whether or not this series converges. Let $f(x) = \frac{1}{x \ln x}$. The function $f(x)$ is decreasing for $x \geq 2$. We must now determine whether or not the following integral converges:

$$\int_2^{\infty} \frac{1}{x \ln x} \, dx = \lim_{R \to \infty} \int_2^R \frac{1}{x \ln x} \, dx$$

Let $u = \ln x$. Then $du = \frac{1}{x} \, dx$ and we get:

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x \ln x} \, dx &= \lim_{R \to \infty} \int_2^R \frac{1}{x \ln x} \, dx \\
&= \lim_{R \to \infty} \int_{\ln 2}^{\ln R} \frac{1}{u} \, du \\
&= \lim_{R \to \infty} \Big[ \ln u \Big]_{\ln 2}^{\ln R} \\
&= \lim_{R \to \infty} \Big[ \ln(\ln R) - \ln(\ln 2) \Big] \\
&= \infty
\end{aligned}$$

Since the integral diverges, the series of absolute values diverges by the Integral Test and the series $\sum_{n=2}^{+\infty} \frac{(-1)^n}{n \ln n}$ is not absolutely convergent.

We now check for conditional convergence using the Leibniz Test. Let $a_n = \frac{1}{n \ln n}$. Then $a_n$ is decreasing for $n \geq 2$ and

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n \ln n} = 0$$

Therefore, the series $\sum_{n=2}^{+\infty} \frac{(-1)^n}{n \ln n}$ is **conditionally convergent**.

**Math 181, Exam 2, Study Guide 3**
**Problem 24 Solution**

24. Determine whether the series $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n n^2}{2^n}$ is absolutely convergent, conditionally convergent, or divergent.

**Solution**: The series is alternating so we first check for absolute convergence by considering the series of absolute values:

$$\sum_{n=1}^{+\infty} \left| \frac{(-1)^n n^2}{2^n} \right| = \sum_{n=1}^{+\infty} \frac{n^2}{2^n}$$

We use the Ratio Test to determine whether the series converges or not.

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} \\
&= \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^2 \\
&= \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right)^2 \\
&= \frac{1}{2}
\end{aligned}$$

Since $\rho = \frac{1}{2} < 1$, the series converges by the Ratio Test. Therefore, the series **converges absolutely**.