Final Exam, Fall 2013

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1. Evaluate $\int x \ln(x)\, dx$.

Solution: We use Integration by Parts. Letting $u = \ln(x)$ and $dv = x\, dx$ yields $du = \frac{1}{x}\, dx$ and $v = \frac{1}{2}x^2$. The Integration by Parts formula is

$$\int u\, dv = uv - \int v\, du.$$

Thus, we have

$$\int x \ln(x)\, dx = \frac{1}{2}x^2 \ln(x) - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\, dx$$

$$\int x \ln(x)\, dx = \frac{1}{2}x^2 \ln(x) - \frac{1}{2}\int x\, dx$$

$$\int x \ln(x)\, dx = \frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + C$$

2. Evaluate the integral $\int e^{2x} \cos(x)\, dx$.

Solution: We use Integration by Parts. Letting $u = e^{2x}$ and $dv = \cos(x)\, dx$ yields $du = 2e^{2x}\, dx$ and $v = \sin(x)$. The Integration by Parts formula is

$$\int u\, dv = uv - \int v\, du.$$

Thus, we have

$$\int e^{2x} \cos(x)\, dx = e^{2x} \sin(x) - \int \sin(x) \cdot 2e^{2x}\, dx$$
$$\int e^{2x} \cos(x)\, dx = e^{2x} \sin(x) - 2 \int e^{2x} \sin(x)\, dx$$

We now use another Integration by Parts. Letting $u = e^{2x}$ and $dv = \sin(x)\, dx$ yields $du = 2e^{2x}\, dx$ and $v = -\cos(x)$. Thus, we have

$$\int e^{2x} \cos(x)\, dx = e^{2x} \sin(x) - 2 \int e^{2x} \sin(x)\, dx$$
$$\int e^{2x} \cos(x)\, dx = e^{2x} \sin(x) - 2 \left[-e^{2x} \cos(x) + 2 \int e^{2x} \cos(x)\, dx \right]$$

$$\int e^{2x}\cos(x)\,dx = e^{2x}\sin(x) - 2\left[-e^{2x}\cos(x) + 2\int e^{2x}\cos(x)\,dx\right]$$
$$\int e^{2x}\cos(x)\,dx = e^{2x}\sin(x) + 2e^{2x}\cos(x) - 4\int e^{2x}\cos(x)\,dx$$

Adding $4\int e^{2x}\cos(x)\,dx$ to both sides of the equation, dividing by 5, and adding an integration constant yields the result:

$$\int e^{2x}\cos(x)\,dx = \frac{1}{5}e^{2x}\sin(x) + \frac{2}{5}e^{2x}\cos(x) + C$$

3. Evaluate the integral $\int \sin^5(x)\,dx$.

Solution: The odd power on $\sin(x)$ indicates that we should rewrite the integrand as $\sin^5(x) = \sin(x)\sin^4(x)$. Then we have

$$\sin^4(x) = (\sin^2(x))^2 = (1 - \cos^2(x))^2$$

Thus, the integral may be rewritten as

$$\int \sin^5(x)\,dx = \int \sin(x)(1 - \cos^2(x))^2\,dx$$

Letting $u = \cos(x)$ and $-du = \sin(x)\,dx$ yields

$$\int \sin^5(x)\,dx = \int -(1 - u^2)^2\,du$$

$$\int \sin^5(x)\,dx = \int (-1 + 2u^2 - u^4)\,du$$

$$\int \sin^5(x)\,dx = -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C$$

$$\int \sin^5(x)\,dx = -\cos(x) + \frac{2}{3}\cos^3(x) - \frac{1}{5}\cos^5(x) + C$$

4. Evaluate the integral $\int \frac{dx}{x^2\sqrt{x^2+1}}$.

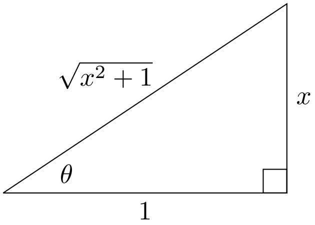
Solution: The integration technique is the trigonometric substitution. Let $x = \tan\theta$. Then $dx = \sec^2\theta$. These substitutions yield the result

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2\theta}{\tan^2\theta\sqrt{\tan^2\theta+1}}$$
$$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2\theta}{\tan^2\theta\cdot\sec\theta}\,d\theta$$
$$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$
$$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

If we let $u = \sin\theta$ and $du = \cos\theta\,d\theta$ then we obtain

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \frac{1}{u^2}\,du = \left(-\frac{1}{u}\right) + C = -\frac{1}{\sin\theta} + C$$

Since $x = \tan\theta$ we have $\tan\theta = \frac{x}{1} = \frac{\text{opposite}}{\text{adjacent}}$. If we draw a right triangle then we take x as the side opposite θ and 1 as the side adjacent to θ.



Thus, $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{\sqrt{x^2+1}}$ where the hypotenuse is obtained using Pythagoras' Theorem. Finally, the integral is

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = -\frac{1}{\sin\theta} + C = -\frac{\sqrt{x^2+1}}{x} + C.$$

5. Evaluate $\int \frac{2x-1}{(x+1)(x+2)}\,dx$.

Solution: We use the method of partial fractions. The integrand may be decomposed as follows:

$$\frac{2x-1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}.$$

After clearing denominators we obtain

$$2x-1 = A(x+2) + B(x+1).$$

When $x = -2$ we have $B = 5$. When $x = -1$ we have $A = -3$. Thus, the integral is

$$\int \frac{2x-1}{(x+1)(x+2)}\,dx = \int \left(\frac{-3}{x+1} + \frac{5}{x+2}\right)\,dx$$

$$\int \frac{2x-1}{(x+1)(x+2)}\,dx = -3\ln|x+1| + 5\ln|x+2| + C$$

6. The region between $y = x$ and $y = x^2$ is rotated about the axis $x = 2$. Compute the volume of the resulting solid.

Solution: The curves intersect when $x^2 = x$. The solutions to the equation are $x = 0$ and $x = 1$.

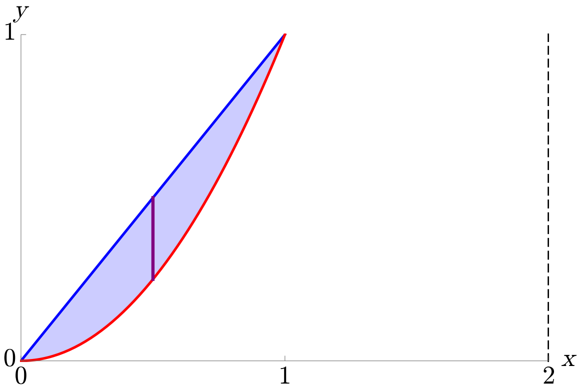
We calculate the volume using shells. The corresponding formula is

$$V = \int_a^b 2\pi(2 - x)(f(x) - g(x))\,dx$$

where $a = 0$, $b = 2$, $f(x) = x$, and $g(x) = x^2$. Thus, the volume is

$$V = \int_0^1 2\pi(2 - x)(x - x^2)\,dx = 2\pi \int_0^1 (2x - 3x^2 + x^3)\,dx$$

$$= 2\pi \left[x^2 - x^3 + \frac{1}{4}x^4\right]_0^1 = \frac{\pi}{2}$$



A vertical cross section (purple line) is at a distance of x from the y-axis. Thus, its distance from the line $x = 2$ is $2 - x$. This is the radius of the shell.

7. Find the arc length of the curve $y = \frac{x^4}{4} + \frac{1}{8x^2}$ on the interval $[1, 2]$.

Solution: The arclength formula is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

We have $\frac{dy}{dx} = x^3 - \frac{1}{4}x^{-3}$. Therefore,

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(x^3 - \frac{1}{4}x^{-3}\right)^2} = \sqrt{1 + x^6 - \frac{1}{2} + \frac{1}{16}x^{-6}}$$

$$= \sqrt{x^6 + \frac{1}{2} + \frac{1}{16}x^{-6}} = \sqrt{\left(x^3 + \frac{1}{4}x^{-3}\right)^2} = x^3 + \frac{1}{4}x^{-3}$$

Therefore, the arclength is

$$L = \int_1^2 \left(x^3 + \frac{1}{4}x^{-3}\right)\, dx = \left[\frac{1}{4}x^4 - \frac{1}{8x^2}\right]_1^2 = \frac{123}{32}$$

8. Compute T_3 for the integral $\int_0^1 x^3\,dx$. Give your answer as a reduced fraction with integer numerator and denominator.

Solution: The Trapezoidal rule with $n = 3$ is

$$T_3 = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right]$$

where $\Delta x = \frac{b-a}{n} = \frac{1-0}{3} = \frac{1}{3}$ and $\{x_0, x_1, x_2, x_3\} = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ and the function is $f(x) = x^3$. Therefore, the value of T_3 is

$$T_3 = \frac{\frac{1}{3}}{2}\left[f(0) + 2f\left(\frac{1}{3}\right) + 2f\left(\frac{2}{3}\right) + f(1)\right]$$

$$T_3 = \frac{1}{6}\left[0 + 2\cdot\left(\frac{1}{3}\right)^3 + 2\cdot\left(\frac{2}{3}\right)^3 + 1^3\right]$$

$$T_3 = \frac{5}{18}$$

9. Evaluate $\int_0^{+\infty} e^{-2x}\,dx$.

Solution: The integral is improper. Thus, we transform it into a limit calculation and evaluate it as follows:

$$\int_0^{+\infty} e^{-2x}\,dx = \lim_{b\to+\infty} \int_0^{b} e^{-2x}\,dx$$

$$\int_0^{+\infty} e^{-2x}\,dx = \lim_{b\to+\infty} \left[-\frac{1}{2}e^{-2x}\right]_0^b$$

$$\int_0^{+\infty} e^{-2x}\,dx = \lim_{b\to+\infty} \left[-\frac{1}{2}e^{-2b} + \frac{1}{2}\right]$$

$$\int_0^{+\infty} e^{-2x}\,dx = \frac{1}{2}$$

10. Determine whether the integral $\int_e^{+\infty} \frac{dx}{x\ln(x)}$ converges or not.

Solution: The integral is improper. Thus, we transform it into a limit calculation:

$$\int_e^{+\infty} \frac{dx}{x\ln(x)} = \lim_{b\to+\infty} \int_e^{b} \frac{dx}{x\ln(x)}$$

We integrate using $u = \ln(x)$ and $du = \frac{1}{x}\,dx$. The limits of integration becomes $u = \ln(e) = 1$ and $u = \ln(b)$. Thus, we have

$$\int_e^{+\infty} \frac{dx}{x\ln(x)} = \lim_{b\to+\infty} \int_1^{\ln(b)} \frac{1}{u}\,du$$

$$\int_e^{+\infty} \frac{dx}{x\ln(x)} = \lim_{b\to+\infty} \Big[\ln|u|\Big]_1^{\ln(b)}$$

$$\int_e^{+\infty} \frac{dx}{x\ln(x)} = \lim_{b\to+\infty} \Big[\ln(\ln(b)) - \ln(1)\Big]$$

$$\int_e^{+\infty} \frac{dx}{x\ln(x)} = +\infty$$

Thus, the integral diverges.

11. Find the limit of the sequence $\left\{ \frac{\sin(\frac{n\pi}{2})}{3^n} \right\}$.

Solution: Because $-1 \leq \sin(\frac{n\pi}{2}) \leq 1$, we have

$$-\frac{1}{3^n} \leq \frac{\sin(\frac{n\pi}{2})}{3^n} \leq \frac{1}{3^n}$$

for all $n \geq 1$. Since

$$\lim_{n\to\infty} -\frac{1}{3^n} = 0 \text{ and } \lim_{n\to\infty} \frac{1}{3^n} = 0$$

by the Squeeze Theorem we have

$$\lim_{n\to\infty} \frac{\sin(\frac{n\pi}{2})}{3^n} = 0$$

12. Find the sum of the series $\sum_{n=3}^{+\infty} \frac{2^n + 7^n}{9^n}$.

Solution: The series may be rewritten as

$$\sum_{n=3}^{+\infty} \frac{2^n + 7^n}{9^n} = \sum_{n=3}^{+\infty} \left(\frac{2}{9}\right)^n + \sum_{n=3}^{+\infty} \left(\frac{7}{9}\right)^n$$

Each of the series on the right hand side above is a geometric series that converges. The sum is

$$\sum_{n=3}^{+\infty} \left(\frac{2}{9}\right)^n + \sum_{n=3}^{+\infty} \left(\frac{7}{9}\right)^n = \frac{(\frac{2}{9})^3}{1 - \frac{2}{9}} + \frac{(\frac{7}{9})^3}{1 - \frac{7}{9}}$$

13. Determine whether the series $\sum_{n=1}^{+\infty} \frac{n}{2n+1}$ converges or not.

Solution: We use the Divergence Test.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}.$$

Since the limit is not zero, the series diverges.

14. Determine whether the series $\displaystyle\sum_{n=1}^{+\infty} \frac{\sin(n)}{n^2}$ converges or not.

Solution: The sequence $a_n = \frac{\sin(n)}{n^2}$ has infinitely many positive and infinitely many negative terms. Thus, we test for absolute convergence by considering the series of absolute values:

$$\sum \left| \frac{\sin(n)}{n^2} \right| = \sum \frac{|\sin(n)|}{n^2}.$$

Since $0 \leq \frac{|\sin(n)|}{n^2} \leq \frac{1}{n^2}$ for all $n \geq 1$ and the series $\sum \frac{1}{n^2}$ is a convergent p-series ($p = 2 > 1$), the series $\sum \frac{|\sin(n)|}{n^2}$ converges by the Comparison Test. Thus, the series $\sum \frac{\sin(n)}{n^2}$ is absolutely convergent and converges.

15. Determine if the series $\sum_{n=1}^{+\infty} \left(\frac{n+1}{3n}\right)^n$ converges or not.

Solution: We use the Root Test.

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left[\left(\frac{n+1}{3n}\right)^n\right]^{1/n} = \lim_{n\to\infty} \frac{n+1}{3n} = \frac{1}{3}$$

Since $\rho = \frac{1}{3} < 1$, the series converges.

16. Find the third degree Taylor *polynomial* of $f(x) = \sqrt{x}$ centered at $a = 4$.

Solution: f and its first three derivatives evaluated at $x = 4$ are

$$\begin{aligned} f(x) &= x^{1/2} & f(4) &= 4^{1/2} = 2 \\ f'(x) &= \frac{1}{2}x^{-1/2} & f'(4) &= \frac{1}{2}4^{-1/2} = \frac{1}{4} \\ f''(x) &= -\frac{1}{4}x^{-3/2} & f''(4) &= -\frac{1}{4}4^{-3/2} = -\frac{1}{32} \\ f'''(x) &= \frac{3}{8}x^{-5/2} & f'''(4) &= \frac{3}{8}4^{-5/2} = \frac{3}{256} \end{aligned}$$

The third order Taylor polynomial of f centered at 4 is

$$p_3(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \frac{f'''(4)}{3!}(x-4)^3$$

$$p_3(x) = 2 + \frac{1}{4}(x-4) + \frac{-\frac{1}{32}}{2!}(x-4)^2 + \frac{\frac{3}{256}}{3!}(x-4)^3$$

$$p_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$$

17. Determine the interval of convergence for the power series $\sum_{k=1}^{+\infty} kx^k$.

(Be sure to check the endpoints of the interval.)

Solution: We use the Ratio Test to find the interval of convergence. Testing for absolute convergence we have

$$r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(k+1)x^{k+1}}{kx^k}\right| = \lim_{k\to\infty} |x| \cdot \frac{k+1}{k} = |x|.$$

According to the Ratio Test, the series will converge when $r = |x| < 1$, i.e. $-1 < x < 1$. However, the test is inconclusive when $r = |x| = 1$, i.e. when $x = -1$ or $x = 1$.

- When $x = 1$, the power series becomes $\sum_{k=1}^{\infty} k$ which diverges.
- When $x = -1$, the power series becomes $\sum_{k=1}^{\infty} k(-1)^k$ which diverges.

Thus, the interval of convergence is $-1 < x < 1$.

18. Find the Maclaurin series expansion of the function $f(x) = \sin(3x^2)$. Write your answer in summation form.

Solution: The Maclaurin series for $\sin(x)$ is

$$\sin(x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

Replacing x with $3x^2$ yields

$$\sin(3x^2) = \sum_{k=0}^{\infty}(-1)^k \frac{(3x^2)^{2k+1}}{(2k+1)!}$$

$$\sin(3x^2) = \sum_{k=0}^{\infty}(-1)^k \frac{3^{2k+1}x^{4k+2}}{(2k+1)!}$$

19. Find the slope of the line tangent to the polar curve $r = 1 + \sin\theta$ at the point $(1, \pi)$.

Solution: Since $x = r\cos\theta$ and $y = r\sin\theta$ we have

$$x = (1 + \sin\theta)\cos\theta, \quad y = (1 + \sin\theta)\sin\theta.$$

The derivatives of x and y with respect to θ are

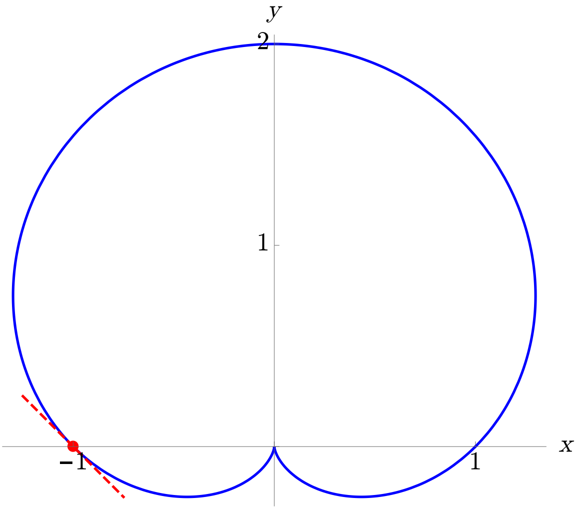
$$\frac{dx}{d\theta} = \cos\theta\cos\theta - (1 + \sin\theta)\sin\theta$$
$$\frac{dy}{d\theta} = \cos\theta\sin\theta + (1 + \sin\theta)\cos\theta$$

The derivatives evaluated at $\theta = \pi$ are

$$\left.\frac{dx}{d\theta}\right|_{\theta=\pi} = 1, \quad \left.\frac{dy}{d\theta}\right|_{\theta=\pi} = -1$$

Thus, the slope of the line tangent to $r = 1 + \sin\theta$ at $(r, \theta) = (1, \pi)$ is

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-1}{1} = -1$$


y
2
1
x
−1
1

20. Find a Cartesian equivalent for the polar equation $r = \dfrac{4}{2\cos\theta - \sin\theta}$ and sketch the curve.

Solution: Multiplying both sides of the equation by the denominator on the left hand side we obtain

$$\begin{aligned} r(2\cos\theta - \sin\theta) &= 4 \\ 2r\cos\theta - r\sin\theta &= 4 \\ 2x - y &= 4 \end{aligned}$$

This is the line $y = 2x - 4$ which has slope 2 and y-intercept -4. The plot is shown below.