**Math 181** **Name (Print):** ______________________
**Fall 2013**
**Hour Exam 2**
**11/08/13**
**Time Limit: 50 Minutes**

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This exam contains 9 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any electronic device including cell phones.

The following rules apply:

- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little to no credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- You must support your answers in all limit problems by a calculation or a brief explanation.
- For each series, you must clearly indicate which test you are using. You must also provide a proper conclusion.

| Problem | Points | Score |
|---|---|---|
| 1 | 30 | |
| 2 | 30 | |
| 3 | 30 | |
| 4 | 30 | |
| 5 | 25 | |
| 6 | 25 | |
| 7 | 30 | |
| Total: | 200 | |

1. (30 points) Let $C$ be the curve given in polar coordinates by $r = \tan\theta$ for $0 \le \theta < \frac{\pi}{2}$.

   (a) Express $C$ as an equation of the form $y = f(x)$.

   (b) Sketch the curve $C$.

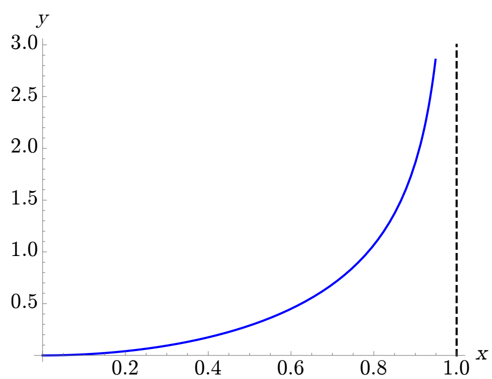
   (c) Find the area of the region bounded by $C$ and the line $\theta = \frac{\pi}{4}$.

*Solution*:

(a) Squaring both sides of the equation yields $r^2 = \tan^2\theta$. We now replace $r^2$ with $x^2 + y^2$ and $\tan\theta$ with $\frac{y}{x}$ and then simplify to obtain:

$$\begin{aligned} r^2 &= \tan^2\theta \\ x^2 + y^2 &= \frac{y^2}{x^2} \\ x^4 + x^2y^2 &= y^2 \\ y^2 - x^2y^2 &= x^4 \\ y^2(1 - x^2) &= x^4 \\ y^2 &= \frac{x^4}{1 - x^2} \\ y &= \frac{x^2}{\sqrt{1 - x^2}} \end{aligned}$$

(b) A sketch of the curve is shown below.




(c) The area of the region bounded by $r = \tan\theta$ and $\theta = \frac{\pi}{4}$ is

$$\begin{aligned} A &= \frac{1}{2}\int_{\alpha}^{\beta} f(\theta)^2\,d\theta \\ A &= \frac{1}{2}\int_{0}^{\pi/4} \tan^2\theta\,d\theta \\ A &= \frac{1}{2}\int_{0}^{\pi/4} (\sec^2\theta - 1)\,d\theta \\ A &= \frac{1}{2}\Big[\tan\theta - \theta\Big]_{0}^{\pi/4} \\ A &= \frac{1}{2}\left(1 - \frac{\pi}{4}\right) \end{aligned}$$

2. (30 points) Decide whether each of the following series converges conditionally, converges absolutely, or diverges.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k+\sqrt{k}}$ (b) $\displaystyle\sum_{m=3}^{\infty} \frac{\sin(m)}{m^3}$ (c) $\displaystyle\sum_{j=1}^{\infty} \frac{(-3)^j}{2^j+3^j}$

*Solution*:

(a) We test for absolute convergence by considering the series of absolute values:

$$\sum \left|\frac{(-1)^k}{k+\sqrt{k}}\right| = \sum \frac{1}{k+\sqrt{k}}.$$

Let $a_k = \frac{1}{k+\sqrt{k}}$. We use the Limit Comparison Test with the series $\sum b_k = \sum \frac{1}{k}$. The value of $L$ in the LCT is

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{1}{k+\sqrt{k}}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{k+\sqrt{k}} = \lim_{k\to\infty} \frac{1}{1+\frac{1}{\sqrt{k}}} = 1.$$

Since $0 < L < \infty$ and $\sum \frac{1}{k}$ diverges ($p$-series with $p = 1 \le 1$), the series $\sum \frac{1}{k+\sqrt{k}}$ diverges by the LCT. Thus, the alternating series is not absolutely convergent.

We now use the Alternating Series Test. The sequence $a_k$ is decreasing for $k \ge 1$ and

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k+\sqrt{k}} = 0.$$

Thus, the series converges and is conditionally convergent since it is not absolutely convergent.

(b) The sequence $a_m = \frac{\sin(m)}{m^3}$ has infinitely many positive and infinitely many negative terms. Thus, we test for absolute convergence by considering the series of absolute values:

$$\sum \left|\frac{\sin(m)}{m^3}\right| = \sum \frac{|\sin(m)|}{m^3}.$$

Since $0 \le \frac{|\sin(m)|}{m^3} \le \frac{1}{m^3}$ for all $m \ge 1$ and the series $\sum \frac{1}{m^3}$ is a convergent $p$-series ($p = 3 > 1$), the series $\sum \frac{|\sin(m)|}{m^3}$ converges by the Comparison Test. Thus, the series $\sum \frac{\sin(m)}{m^3}$ is absolutely convergent.

(c) Let $a_j = \frac{(-3)^j}{2^j+3^j}$. Then,

$$\lim_{j\to\infty} a_j = \lim_{j\to\infty} \frac{(-3)^j}{2^j+3^j} = \lim_{j\to\infty} (-1)^j \frac{3^j}{2^j+3^j} = \lim_{j\to\infty} (-1)^j \frac{1}{(\frac{2}{3})^j+1}$$

which does not exist. [The expression $\frac{1}{(\frac{2}{3})^j+1}$ tends to 1 as $j \to \infty$ but $(-1)^j$ alternates between $-1$ and 1.] Since the limit is not zero, the series diverges by the Divergence Test.

3. (30 points) Let $R$ be the region bounded by the curves $y = x^2$ and $y = 2x$.

   (a) Determine where these curves cross and sketch the region $R$.

   (b) Find the area of $R$.

   (c) Find the volume of the solid obtained by revolving $R$ around the vertical line $x = 3$.

*Solution*:

(a) The curves intersect when $x^2 = 2x$. The solutions to the equation are $x = 0$ and $x = 2$. The corresponding $y$-values are $y = 0$ and $y = 4$, respectively.

(b) The area of $R$ is

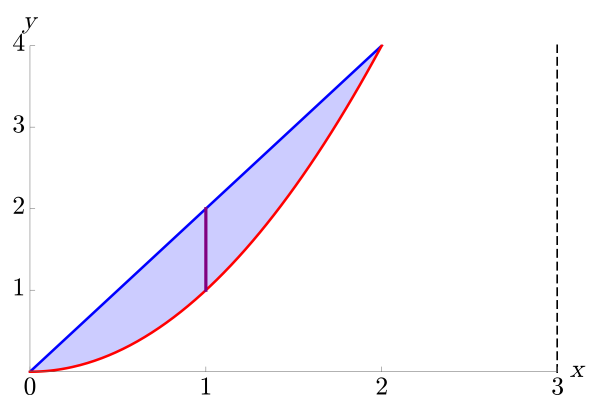
$$A = \int_0^2 (2x - x^2)\, dx = \left[x^2 - \frac{1}{3}x^3\right]_0^2 = 2^2 - \frac{1}{3}2^3 = \frac{4}{3}.$$

(c) We calculate the volume using shells. The corresponding formula is

$$V = \int_a^b 2\pi(3 - x)(f(x) - g(x))\, dx$$

where $a = 0$, $b = 2$, $f(x) = 2x$, and $g(x) = x^2$. Thus, the volume is

$$V = \int_0^2 2\pi(3-x)(2x-x^2)\, dx = 2\pi \int_0^2 (6x-5x^2+x^3)\, dx = 2\pi \left[3x^2 - \frac{5}{3}x^3 + \frac{1}{4}x^4\right]_0^2 = \frac{16\pi}{3}$$



4. (30 points) Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{2(x+1)^n}{5^n n^2}$.

*Solution*: We use the Ratio Test to find the interval of convergence. Testing for absolute convergence we have

$$\begin{aligned}
r &= \lim_{n\to\infty} \left| a_{n+1} \cdot \frac{1}{a_n} \right| \\
r &= \lim_{n\to\infty} \left| \frac{2(x+1)^{n+1}}{5^{n+1}(n+1)^2} \cdot \frac{5^n n^2}{2(x+1)^n} \right| \\
r &= \lim_{n\to\infty} \frac{1}{5}|x+1| \cdot \left( \frac{n}{n+1} \right)^2 \\
r &= \frac{1}{5}|x+1| \underbrace{\lim_{n\to\infty} \left( \frac{n}{n+1} \right)^2}_{=1} \\
r &= \frac{1}{5}|x+1|
\end{aligned}$$

According to the Ratio Test, the series will converge when $r = \frac{1}{5}|x+1| < 1$, i.e.

$$\begin{aligned}
\frac{1}{5}|x+1| &< 1 \\
|x+1| &< 5 \\
-5 < x+1 &< 5 \\
-6 < x &< 4
\end{aligned}$$

However, the test is inconclusive when $r = \frac{1}{5}|x+1| = 1$, i.e. when $x = -6$ or $x = 4$.

- When $x = -6$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{2(-6+1)^n}{5^n n^2} = \sum_{n=1}^{\infty} \frac{2(-5)^n}{5^n n^2} = \sum_{n=1}^{\infty} 2(-1)^n \frac{1}{n^2}.$$

  This is an absolutely convergent series because the series of absolute values $\sum 2 \cdot \frac{1}{n^2}$ is a convergent $p$-series.

- When $x = 4$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{2(4+1)^n}{5^n n^2} = \sum_{n=1}^{\infty} \frac{2(5)^n}{5^n n^2} = \sum_{n=1}^{\infty} 2 \cdot \frac{1}{n^2}.$$

  This is a convergent $p$-series.

Thus, the interval of convergence is $-6 \le x \le 4$.

5. (25 points) Find the third order Taylor polynomial for $f(x) = \sqrt{x}$ centered at $x = \frac{1}{4}$.

*Solution*: $f$ and its first three derivatives evaluated at $x = \frac{1}{4}$ are

$$
\begin{aligned}
f(x) &= x^{1/2} & f(\tfrac{1}{4}) &= (\tfrac{1}{4})^{1/2} = \tfrac{1}{2} \\
f'(x) &= \tfrac{1}{2}x^{-1/2} & f'(\tfrac{1}{4}) &= \tfrac{1}{2}(\tfrac{1}{4})^{-1/2} = 1 \\
f''(x) &= -\tfrac{1}{4}x^{-3/2} & f''(\tfrac{1}{4}) &= -\tfrac{1}{4}(\tfrac{1}{4})^{-3/2} = -2 \\
f'''(x) &= \tfrac{3}{8}x^{-5/2} & f'''(\tfrac{1}{4}) &= \tfrac{3}{8}(\tfrac{1}{4})^{-5/2} = 12
\end{aligned}
$$

The third order Taylor polynomial of $f$ centered at $\frac{1}{4}$ is

$$
\begin{aligned}
p_3(x) &= f(\tfrac{1}{4}) + f'(\tfrac{1}{4})(x - \tfrac{1}{4}) + \tfrac{f''(\frac{1}{4})}{2!}(x - \tfrac{1}{4})^2 + \tfrac{f'''(\frac{1}{4})}{3!}(x - \tfrac{1}{4})^3 \\
p_3(x) &= \tfrac{1}{2} + 1 \cdot (x - \tfrac{1}{4}) + \tfrac{-2}{2!}(x - \tfrac{1}{4})^2 + \tfrac{12}{3!}(x - \tfrac{1}{4})^3 \\
p_3(x) &= \tfrac{1}{2} + (x - \tfrac{1}{4}) - (x - \tfrac{1}{4})^2 + 2(x - \tfrac{1}{4})^3
\end{aligned}
$$

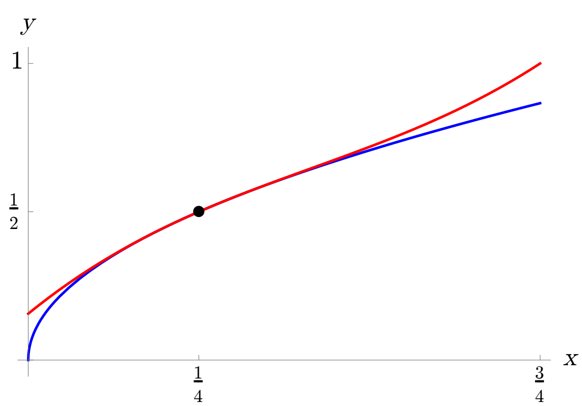



Figure 1: Plots of $f(x) = \sqrt{x}$ (blue) and $p_3(x)$ (red)

6. (25 points) Let $C$ be the parametrized curve $x = 2\sin(t)$, $y = \sin(2t)$ for $0 \le t \le 2\pi$.

   (a) Find an expression for $\frac{dy}{dx}$ as a function of $t$.

   (b) Find all the points of $C$ where the tangent line is horizontal and all the points where it is vertical.

*Solution*:

(a) The derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are

$$\frac{dx}{dt} = 2\cos(t), \qquad \frac{dy}{dt} = 2\cos(2t).$$

Thus, the derivative $\frac{dy}{dx}$ in terms of $t$ is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos(2t)}{2\cos(t)} = \frac{\cos(2t)}{\cos(t)}.$$

(b) The tangent line is horizontal when the derivative is zero, i.e. when $\cos(2t) = 0$. The solutions on the interval $0 \le t \le 2\pi$ are

$$t = \frac{\pi}{4}, \ \frac{3\pi}{4}, \ \frac{5\pi}{4}, \ \frac{7\pi}{4}.$$

The corresponding points on the curve $C$ are

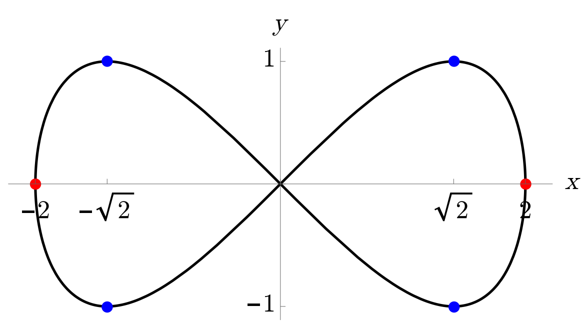
$$(\sqrt{2}, 1), \ (\sqrt{2}, -1), \ (-\sqrt{2}, 1), \ (-\sqrt{2}, -1).$$

The tangent line is vertical when the derivative is undefined, i.e. when $\cos(t) = 0$. The solutions on the interval $0 \le t \le 2\pi$ are

$$t = \frac{\pi}{2}, \ \frac{3\pi}{2}.$$

The corresponding points on the curve $C$ are

$$(2, 0), \ (-2, 0).$$



7. (30 points) Evaluate the following integrals:

(a) $\displaystyle\int_8^{14} \frac{12}{x^2+8x-20}\,dx$

(b) $\displaystyle\int \frac{dx}{x^2\sqrt{9x^2+4}}$

(c) $\displaystyle\int_1^e \frac{\ln(x)}{x^3}\,dx$

*Solution*:

(a) The integration technique is partial fractions. The integrand may be decomposed as follows:

$$\frac{12}{x^2+8x-20} = \frac{A}{x+10} + \frac{B}{x-2}.$$

After clearing denominators we obtain

$$12 = A(x-2) + B(x+10).$$

When $x = 2$ we have $B = 1$ and when $x = -10$ we have $A = -1$. Thus, the integral becomes

$$\begin{aligned}
\int_8^{14} \frac{12}{x^2+8x-20}\,dx &= \int_8^{14} \left(\frac{-1}{x+10} + \frac{1}{x-2}\right) dx \\
&= \Big[-\ln|x+10| + \ln|x-2|\Big]_8^{14} \\
&= \Big[-\ln(24) + \ln(12)\Big] - \Big[-\ln(18) + \ln(6)\Big] \\
&= \ln(12) + \ln(18) - \ln(24) - \ln(6) \\
&= \ln\left(\frac{12\cdot 18}{24\cdot 6}\right) \\
&= \ln\left(\frac{3}{2}\right)
\end{aligned}$$

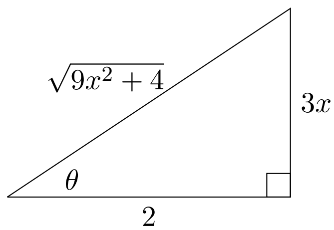
(b) The integration technique is the trigonometric substitution. Let $x = \frac{2}{3}\tan\theta$. Then $dx = \frac{2}{3}\sec^2\theta$. These substitutions yield the result

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{9x^2+4}} &= \int \frac{\frac{2}{3}\sec^2\theta}{(\frac{2}{3}\tan\theta)^2\sqrt{9(\frac{2}{3}\tan\theta)^2+4}} \\
&= \int \frac{\frac{2}{3}\sec^2\theta}{\frac{4}{9}\tan^2\theta\sqrt{4\tan^2\theta+4}}\,d\theta \\
&= \int \frac{\frac{2}{3}\sec^2\theta}{\frac{4}{9}\tan^2\theta\cdot 2\sec\theta}\,d\theta \\
&= \frac{3}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta \\
&= \frac{3}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta
\end{aligned}$$

If we let $u = \sin\theta$ and $du = \cos\theta\, d\theta$ then we obtain

$$\int \frac{dx}{x^2\sqrt{9x^2+4}} = \frac{3}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta = \frac{3}{4}\int \frac{1}{u^2}\, du = \frac{3}{4}\left(-\frac{1}{u}\right) + C = -\frac{3}{4\sin\theta} + C$$

Since $x = \frac{2}{3}\tan\theta$ we have $\tan\theta = \frac{3x}{2} = \frac{\text{opposite}}{\text{adjacent}}$. If we draw a right triangle then we take $3x$ as the side opposite $\theta$ and 2 as the side adjacent to $\theta$.




Thus, $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3x}{\sqrt{9x^2+4}}$ where the hypotenuse is obtained using Pythagoras' Theorem. Finally, the integral is

$$\int \frac{dx}{x^2\sqrt{9x^2+4}} = -\frac{3}{4\sin\theta} + C = -\frac{\sqrt{9x^2+4}}{4x} + C.$$

(c) We begin by rewriting the integral as

$$\int_1^e \frac{\ln(x)}{x^3}\, dx = \int_1^e x^{-3}\ln(x)\, dx.$$

Letting $u = \ln(x)$ and $dv = x^{-3}\, dx$ yields $du = \frac{1}{x}\, dx$ and $v = -\frac{1}{2}x^{-2}$. The integration by parts formula yields:

$$\begin{aligned}\int u\, dv &= uv - \int v\, du \\ \int x^{-3}\ln(x)\, dx &= -\frac{1}{2}x^{-2}\ln(x) - \int \left(-\frac{1}{2}x^{-2}\right)\left(\frac{1}{x}\right)\, dx \\ &= -\frac{1}{2}x^{-2}\ln(x) + \frac{1}{2}\int x^{-3}\, dx \\ &= -\frac{1}{2}x^{-2}\ln(x) + \frac{1}{2}\left[-\frac{1}{2}x^{-2}\right] \\ &= -\frac{1}{2}x^{-2}\ln(x) - \frac{1}{4}x^{-2}\end{aligned}$$

The value of the definite integral on the interval $[1, e]$ is

$$\int_1^e \frac{\ln(x)}{x^3}\, dx = \left[-\frac{1}{2}x^{-2}\ln(x) - \frac{1}{4}x^{-2}\right]_1^e = -\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2} + \frac{1}{4} = -\frac{3}{4}e^{-2} + \frac{1}{4}$$