Math 210, Exam 1, Fall 2008
Problem 1 Solution

1. Let $P = (-1, 4, 1)$, $Q = (1, 2, 9)$, and $R = (5, 10, 1)$.

   (a) Find the lengths of the sides PQ and PR of the triangle PQR.

   (b) Find the interior angle at the vertex P of the triangle PQR.

   (c) Find the equation of the plane containing P, Q, and R.

   (d) Find the area of the triangle PQR.

Solution:

(a) The vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are obtained by subtracting coordinates as follows:

$$\overrightarrow{PQ} = \langle 1-(-1), 2-4, 9-1\rangle = \langle 2, -2, 8\rangle$$
$$\overrightarrow{PR} = \langle 5-(-1), 10-4, 1-1\rangle = \langle 6, 6, 0\rangle$$

The lengths of sides PQ and PR are the magnitudes of the above vectors:

$$\left|\left|\overrightarrow{PQ}\right|\right| = \sqrt{2^2 + (-2)^2 + 8^2} = \sqrt{72} = \boxed{6\sqrt{2}}$$
$$\left|\left|\overrightarrow{PR}\right|\right| = \sqrt{6^2 + 6^2 + 0^2} = \sqrt{72} = \boxed{6\sqrt{2}}$$

(b) Use the dot product to determine the angle:

$$\cos\theta = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{||\overrightarrow{PQ}||||\overrightarrow{PR}||}$$
$$\cos\theta = \frac{(2)(6) + (-2)(6) + (8)(0)}{(6\sqrt{2})(6\sqrt{2})}$$
$$\cos\theta = 0$$

Therefore, the angle is $\boxed{\theta = \frac{\pi}{2}}$.

(c) A vector perpendicular to the plane is the cross product of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ which both lie in the plane.

$$\begin{aligned}
\vec{\mathbf{n}} &= \overrightarrow{PQ} \times \overrightarrow{PR} \\
\vec{\mathbf{n}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -2 & 8 \\ 6 & 6 & 0 \end{vmatrix} \\
\vec{\mathbf{n}} &= \hat{\mathbf{i}} \begin{vmatrix} -2 & 8 \\ 6 & 0 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} 2 & 8 \\ 6 & 0 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} 2 & -2 \\ 6 & 6 \end{vmatrix} \\
\vec{\mathbf{n}} &= \hat{\mathbf{i}}\,[(-2)(0) - (8)(6)] - \hat{\mathbf{j}}\,[(2)(0) - (8)(6)] + \hat{\mathbf{k}}\,[(2)(6) - (-2)(6)] \\
\vec{\mathbf{n}} &= -48\,\hat{\mathbf{i}} + 48\,\hat{\mathbf{j}} + 24\,\hat{\mathbf{k}} \\
\vec{\mathbf{n}} &= \langle -48, 48, 24 \rangle
\end{aligned}$$

Using $P = (-1, 4, 1)$ as a point on the plane, we have:

$$\boxed{-48(x+1) + 48(y-4) + 24(z-1) = 0}$$

(d) The area of the triangle is half the magnitude of the cross product of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, which represents the area of the parallelogram spanned by the two vectors:

$$\begin{aligned}
A &= \frac{1}{2}\left|\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|\right| \\
A &= \frac{1}{2}\sqrt{(-48)^2 + 48^2 + 24^2} \\
A &= \frac{1}{2}(72) \\
&\boxed{A = 36}
\end{aligned}$$

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Problem 2 Solution

2. Consider a particle traveling along the curve $\vec{\mathbf{r}}(t) = \langle t, 2e^t, e^{2t} \rangle$.

(a) Calculate the position, velocity, speed and acceleration of the particle at $t = 1$.

(b) What is the length of $\vec{\mathbf{r}}(t)$ between $t = 1$ and $t = 2$?

(c) Find an equation for the tangent line to $\vec{\mathbf{r}}(t)$ at $t = 0$.

(d) Calculate the curvature of $\vec{\mathbf{r}}(t)$ at $t = 0$.

Solution:

(a) The velocity and acceleration are:

$$\begin{aligned} \vec{\mathbf{v}}(t) &= \vec{\mathbf{r}}'(t) = \left\langle 1, 2e^t, 2e^{2t} \right\rangle \\ \vec{\mathbf{a}}(t) &= \vec{\mathbf{r}}''(t) = \left\langle 0, 2e^t, 4e^{2t} \right\rangle \end{aligned}$$

At $t = 1$ we have:

$$\begin{aligned} \vec{\mathbf{r}}(1) &= \left\langle 1, 2e, e^2 \right\rangle \\ \vec{\mathbf{v}}(1) &= \left\langle 1, 2e, 2e^2 \right\rangle \\ \vec{\mathbf{a}}(1) &= \left\langle 0, 2e, 4e^2 \right\rangle \\ v(1) &= \sqrt{1^2 + (2e)^2 + (2e^2)^2} \\ &= \sqrt{1 + 4e^2 + 4e^4} \end{aligned}$$

(b) The length of the curve is:

$$\begin{aligned} L &= \int_1^2 \left|\left|\vec{\mathbf{r}}'(t)\right|\right| \, dt \\ L &= \int_1^2 \sqrt{1 + 4e^{2t} + 4e^{4t}} \, dt \\ L &= \int_1^2 \sqrt{(1 + 2e^{2t})^2} \, dt \\ L &= \int_1^2 (1 + 2e^{2t}) \, dt \\ L &= \left[t + e^{2t} \right]_1^2 \\ L &= (2 + e^4) - (1 + e^2) \\ L &= \boxed{1 + e^4 - e^2} \end{aligned}$$

(c) The tangent line equation is:

$$\overrightarrow{\mathbf{L}}(t) = \overrightarrow{\mathbf{r}}(0) + t\,\overrightarrow{\mathbf{r}}'(0)$$

$$\boxed{\overrightarrow{\mathbf{L}}(t) = \langle 0, 2, 1\rangle + t\,\langle 1, 2, 2\rangle}$$

(d) The curvature at $t = 0$ is:

$$\kappa(0) = \frac{||\overrightarrow{\mathbf{r}}'(0) \times \overrightarrow{\mathbf{r}}''(0)||}{||\overrightarrow{\mathbf{r}}'(0)||^3}$$

$$\kappa(0) = \frac{||\,\langle 1, 2, 2\rangle \times \langle 0, 2, 4\rangle\,||}{||\,\langle 1, 2, 2\rangle\,||^3}$$

$$\kappa(0) = \frac{||\,\langle 4, -4, 2\rangle\,||}{3^3}$$

$$\kappa(0) = \frac{6}{27}$$

$$\boxed{\kappa(0) = \frac{2}{9}}$$

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Problem 3 Solution

3. Evaluate the limits or determine they do not exist:

(a) $\displaystyle\lim_{(x,y)\to(3,4)} \frac{y-3x}{x^2+y^2}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2+y^2}{x^2+y^2}$

Solution:

(a) The function $f(x,y) = \dfrac{y-3x}{x^2+y^2}$ is continuous at $(3,4)$. Thus, we can evaluate the limit using substitution:

$$\lim_{(x,y)\to(3,4)} \frac{y-3x}{x^2+y^2} = \frac{4-3(3)}{3^2+4^2} = \frac{-5}{25} = \boxed{-\frac{1}{5}}$$

(b) We show that the limit does not exist by computing the limit of $f(x,y)$ along two different paths that approach $(0,0)$.

(i) For the first path we approach $(0,0)$ along the x-axis from the right. In this case, we have $y = 0$ and $x \to 0^+$. The limit is then:

$$\lim_{(x,y)\to(0,0)} \frac{2x^2+y^2}{x^2+y^2} = \lim_{x\to 0^+} \frac{2x^2+0^2}{x^2+0^2} = 2$$

(ii) For the second path we approach $(0,0)$ along the y-axis from above. In this case, we have $x = 0$ and $y \to 0^+$. The limit is then:

$$\lim_{(x,y)\to(0,0)} \frac{2x^2+y^2}{x^2+y^2} = \lim_{y\to 0^+} \frac{2(0)^2+y^2}{0^2+y^2} = 1$$

Since we get two different limits, the limit **does not exist**.