Math 210, Exam 1, Fall 2010
Problem 1 Solution

1. Do the following computations.

(a) Compute $\langle 1,2,3\rangle \cdot \langle -2,0,1\rangle$.

(b) Compute $\langle 1,-1,3\rangle \times \langle -2,-3,1\rangle$.

(c) Find a normal vector to the plane described by $7x+2y-3z$.

(d) Determine if the equations $x-y+2z=1$ and $-x+y-2z=3$ describe parallel planes, and give a reason.

(e) If $P=(4,2,-3)$ and $Q=(2,1,5)$, express the vector $\overrightarrow{PQ}$ in terms of the standard unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$.

Solution:

(a) $\langle 1,2,3\rangle \cdot \langle -2,0,1\rangle = (1)(-2)+(2)(0)+(3)(1) = \boxed{1}$

(b) $\langle 1,-1,3\rangle \times \langle -2,-3,1\rangle = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 3 \\ -2 & -3 & 1 \end{vmatrix} = \boxed{\langle 8,-7,-5\rangle}$

(c) Note that the $7x+2y-3z$ is missing an equal sign and a number on the right hand side of the equal sign. In any case, a vector normal to the plane is $\boxed{\overrightarrow{\mathbf{n}} = \langle 7,2,-3\rangle}$.

(d) The vectors normal to the planes are $\overrightarrow{\mathbf{n}}_1 = \langle 1,-1,2\rangle$ and $\overrightarrow{\mathbf{n}}_2 = \langle -1,1,-2\rangle$, respectively. The vectors are parallel because they are scalar multiples of one another. In fact, $\overrightarrow{\mathbf{n}}_1 = -\overrightarrow{\mathbf{n}}_2$. Thus, the planes are parallel to each other.

(e) $\overrightarrow{PQ} = \langle 2-4, 1-2, 5-(-3)\rangle = \langle -2,-1,8\rangle = \boxed{-2\,\hat{\mathbf{i}} - \hat{\mathbf{j}} + 8\,\hat{\mathbf{k}}}$

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Problem 2 Solution

2. Consider the three points $P = (2, -1, 3)$, $Q = (2, 1, -2)$, and $R = (1, 1, 0)$ in $\mathbb{R}^3$.

   (a) Find an equation for the plane which contains P, Q and R.

   (b) Find the area of the triangle with vertices at P, Q and R.

Solution:

(a) A vector perpendicular to the plane is the cross product of $\overrightarrow{PQ} = \langle 0, 2, -5 \rangle$ and $\overrightarrow{QR} = \langle -1, 0, 2 \rangle$ which both lie in the plane.

$$\overrightarrow{\mathbf{n}} = \overrightarrow{PQ} \times \overrightarrow{QR}$$

$$\overrightarrow{\mathbf{n}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 2 & -5 \\ -1 & 0 & 2 \end{vmatrix}$$

$$\overrightarrow{\mathbf{n}} = \hat{\mathbf{i}} \begin{vmatrix} 2 & -5 \\ 0 & 2 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} 0 & -5 \\ -1 & 2 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix}$$

$$\overrightarrow{\mathbf{n}} = \hat{\mathbf{i}}\,[(2)(2) - (0)(-5)] - \hat{\mathbf{j}}\,[(0)(2) - (-1)(-5)] + \hat{\mathbf{k}}\,[(0)(0) - (-1)(2)]$$

$$\overrightarrow{\mathbf{n}} = 4\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}$$

$$\overrightarrow{\mathbf{n}} = \langle 4, 5, 2 \rangle$$

Using $P = (2, -1, 3)$ as a point on the plane, we have:

$$\boxed{4(x-2) + 5(y+1) + 2(z-3) = 0}$$

(b) The area of the triangle is half the magnitude of the cross product of $\overrightarrow{PQ}$ and $\overrightarrow{QR}$, which represents the area of the parallelogram spanned by the two vectors:

$$A = \frac{1}{2}\left\|\overrightarrow{PQ} \times \overrightarrow{QR}\right\|$$

$$A = \frac{1}{2}\sqrt{4^2 + 5^2 + 2^2}$$

$$A = \frac{1}{2}\sqrt{45}$$

$$\boxed{A = \frac{3\sqrt{5}}{2}}$$

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Problem 3 Solution

3. Let c be the curve given by $\vec{\mathbf{c}}(t) = \langle \cos 2t, 3t - 1, \sin 2t \rangle$.

(a) Find parametric equations for the tangent line to c at $t = \dfrac{\pi}{4}$.

(b) Find the length of the curve c between $t = -\pi$ and $t = \pi$.

(c) Find the curvature of c at $t = 0$.

Solution: We need the first two derivatives of $\vec{\mathbf{c}}(t)$.

$$\vec{\mathbf{c}}'(t) = \langle -2\sin(2t), 3, 2\cos(2t) \rangle$$
$$\vec{\mathbf{c}}''(t) = \langle -4\cos(2t), 0, -4\sin(2t) \rangle$$

(a) The vector form of the tangent line at $t = \frac{\pi}{4}$ is:

$$\vec{\mathbf{L}}(t) = \vec{\mathbf{c}}\left(\frac{\pi}{4}\right) + t\,\vec{\mathbf{c}}'\left(\frac{\pi}{4}\right)$$

Evaluating $\vec{\mathbf{c}}(t)$ and $\vec{\mathbf{c}}'(t)$ at $t = \frac{\pi}{4}$ we have:

$$\vec{\mathbf{c}}\left(\frac{\pi}{4}\right) = \left\langle \cos\frac{\pi}{2}, 3\left(\frac{\pi}{4}\right) - 1, \sin\frac{\pi}{2} \right\rangle = \left\langle 0, \frac{3\pi}{4} - 1, 1 \right\rangle$$
$$\vec{\mathbf{c}}'\left(\frac{\pi}{4}\right) = \left\langle -2\sin\frac{\pi}{2}, 3, 2\cos\frac{\pi}{2} \right\rangle = \langle -2, 3, 0 \rangle$$

At $t = \frac{\pi}{4}$, we have $\vec{\mathbf{c}}(\frac{\pi}{4}) = \left\langle \cos\frac{\pi}{2}, 3(\frac{\pi}{4}) - 1, \sin\frac{\pi}{2} \right\rangle = \left\langle 0, \frac{3\pi}{4} - 1, 1 \right\rangle$. Therefore, the vector form of the tangent line is:

$$\vec{\mathbf{L}}(t) = \left\langle 0, \frac{3\pi}{4} - 1, 1 \right\rangle + t\,\langle -2, 3, 0 \rangle$$

and the corresponding parametric equations are:

$$\boxed{x = -2t, \qquad y = \frac{3\pi}{4} - 1 + 3t, \qquad z = 1}$$

(b) The length of the curve is:

$$L = \int_{-\pi}^{\pi} ||\vec{\mathbf{c}}'(t)||\; dt$$
$$L = \int_{-\pi}^{\pi} ||\langle -2\sin(2t), 3, 2\cos(2t) \rangle||\; dt$$
$$L = \int_{-\pi}^{\pi} \sqrt{4\sin^2(2t) + 9 + 4\cos^2(2t)}\, dt$$
$$L = \int_{-\pi}^{\pi} \sqrt{4 + 9}\, dt$$
$$\boxed{L = 2\pi\sqrt{13}}$$

(c) The curvature at $t = 0$ is:

$$\kappa(0) = \frac{\left|\left|\vec{\mathbf{c}}'(0) \times \vec{\mathbf{c}}''(0)\right|\right|}{\left|\left|\vec{\mathbf{c}}'(0)\right|\right|^3}$$

$$\kappa(0) = \frac{||\langle 0, 3, 2\rangle \times \langle -4, 0, 0\rangle||}{||\langle 0, 3, 2\rangle||^3}$$

$$\kappa(0) = \frac{||\langle 0, -8, 12\rangle||}{||\langle 0, 3, 2\rangle||^3}$$

$$\kappa(0) = \frac{4\sqrt{13}}{(\sqrt{13})^3}$$

$$\boxed{\kappa(0) = \frac{4}{13}}$$

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Problem 4 Solution

4. Find an equation for the tangent plane to the surface $x^2 + 2y^2 - z^2 = 12$ at the point $(2, 2, 2)$.

Solution: First, we note that there is a mistake in the problem. The point $(2, 2, 2)$ is not on the surface. To rectify this error, we change the equation of the surface to

$$x^2 + 2y^2 - z^2 = 8$$

Let $F(x, y, z) = x^2 + 2y^2 - z^2$. An equation for the tangent plane is:

$$F_x(2, 2, 2)(x - 2) + F_y(2, 2, 2)(y - 2) + F_z(2, 2, 2)(z - 2) = 0$$

The partial derivatives of F are:

$$\begin{aligned} F_x &= 2x \\ F_y &= 4y \\ F_z &= -2z \end{aligned}$$

Evaluating at $(2, 2, 2)$ we get:

$$F_x(2, 2, 2) = 4, \qquad F_y(2, 2, 2) = 8, \qquad F_z(2, 2, 2) = -4$$

Thus, an equation for the tangent plane is:

$$\boxed{4(x - 2) + 8(y - 2) - 4(z - 2) = 0}$$

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Problem 5 Solution

5. Find the linearization of the function $f(x, y) = x\cos(\pi y) + ye^x$ at the point $(1, 1)$.

Solution: The linearization of f at $(1, 1)$ has the equation:

$$L(x, y) = f(1, 1) + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1)$$

The partial derivatives of f are:

$$f_x = \cos(\pi y) + ye^x$$
$$f_y = -\pi x\sin(\pi y) + e^x$$

Evaluating f and the partial derivatives at $(1, 1)$ we get:

$$f(1, 1) = -1 + e, \qquad f_x(1, 1) = -1 + e, \qquad f_y(1, 1) = e$$

Thus, the linearization is:

$$\boxed{L(x, y) = -1 + e + (-1 + e)(x - 1) + e(y - 1)}$$

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Problem 6 Solution

6. Let $f(x,y) = \dfrac{x^2}{x^2+y^2}$. Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Solution: The function $f(x,y) = \dfrac{x^2}{x^2+y^2}$ is not continuous at $(0,0)$ as the point is not in the domain of f. If the limit exists, the value of the limit should be independent of the path taken to $(0,0)$. Let's choose Path 1 to be the path $y=0$, $x \to 0^+$. The limit along this path is:

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{x\to 0^+} \frac{x^2}{x^2+0^2} = \lim_{x\to 0^+} \frac{x^2}{x^2} = 1$$

Let's choose Path 2 to be the path $x=0$, $y \to 0^+$. The limit along this path is:

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{y\to 0^+} \frac{0^2}{0^2+y^2} = \lim_{y\to 0^+} \frac{0}{y^2} = 0$$

Thus, since we get two different limits along two different paths to $(0,0)$, the limit **does not exist**.