Math 210, Exam 1, Fall 2011
Problem 1 Solution

1. Consider the three points $P = (5, 2, -1)$, $Q = (1, 4, 1)$, $R = (1, 2, 3)$ in $\mathbb{R}^3$.

   (a) Find an equation for the plane which contains P, Q and R.

   (b) Find the area of the triangle with vertices at P, Q and R.

   (c) Find the angle between PQ and PR.

Solution:

(a) A vector perpendicular to the plane is the cross product of $\overrightarrow{PQ} = \langle -4, 2, 2\rangle$ and $\overrightarrow{QR} = \langle 0, -2, 2\rangle$ which both lie in the plane.

$$\begin{aligned}
\overrightarrow{\mathbf{n}} &= \overrightarrow{PQ} \times \overrightarrow{QR} \\
\overrightarrow{\mathbf{n}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -4 & 2 & 2 \\ 0 & -2 & 2 \end{vmatrix} \\
\overrightarrow{\mathbf{n}} &= \hat{\mathbf{i}} \begin{vmatrix} 2 & 2 \\ -2 & 2 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} -4 & 2 \\ 0 & 2 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} -4 & 2 \\ 0 & -2 \end{vmatrix} \\
\overrightarrow{\mathbf{n}} &= \hat{\mathbf{i}}\,[(2)(2) - (2)(-2)] - \hat{\mathbf{j}}\,[(-4)(2) - (2)(0)] + \hat{\mathbf{k}}\,[(-4)(-2) - (2)(0)] \\
\overrightarrow{\mathbf{n}} &= 8\,\hat{\mathbf{i}} + 8\,\hat{\mathbf{j}} + 8\,\hat{\mathbf{k}} \\
\overrightarrow{\mathbf{n}} &= \langle 8, 8, 8\rangle
\end{aligned}$$

Using $P = (5, 2, -1)$ as a point on the plane, we have:

$$\boxed{8(x-5) + 8(y-2) + 8(z+1) = 0}$$

(b) The area of the triangle is half the magnitude of the cross product of $\overrightarrow{PQ}$ and $\overrightarrow{QR}$, which represents the area of the parallelogram spanned by the two vectors:

$$\begin{aligned}
A &= \frac{1}{2}\left|\left|\overrightarrow{PQ} \times \overrightarrow{QR}\right|\right| \\
A &= \frac{1}{2}\sqrt{8^2 + 8^2 + 8^2} \\
&\boxed{A = 4\sqrt{3}}
\end{aligned}$$

(c) The angle θ between the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ can be found using the dot product. That is,

$$\cos\theta = \frac{\overrightarrow{PQ} \bullet \overrightarrow{PR}}{\left|\left|\overrightarrow{PQ}\right|\right| \left|\left|\overrightarrow{PR}\right|\right|}$$

$$\cos\theta = \frac{\langle -4, 2, 2\rangle \bullet \langle -4, 0, 4\rangle}{||\langle -4, 2, 2\rangle||\, ||\langle -4, 0, 4\rangle||}$$

$$\cos\theta = \frac{(-4)(-4) + (2)(0) + (2)(4)}{\sqrt{(-4)^2 + 2^2 + 2^2}\sqrt{(-4)^2 + 0^2 + 4^2}}$$

$$\cos\theta = \frac{24}{\sqrt{24}\sqrt{32}}$$

$$\cos\theta = \frac{\sqrt{3}}{2}$$

$$\boxed{\theta = \frac{\pi}{6}}$$

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Problem 2 Solution

2. A particle moves along the space curve $\overrightarrow{\mathbf{r}}(t) = \cos(2t)\,\hat{\mathbf{i}} + (3t-1)\,\hat{\mathbf{j}} + \sin(2t)\,\hat{\mathbf{k}}$.

   (a) Find the velocity, speed, and acceleration of the particle (as functions of t).

   (b) Find the principal unit normal vector at $t = 0$.

Solution:

(a) The velocity, speed, and acceleration of the particle are:

$$\begin{aligned} \text{velocity} &= \overrightarrow{\mathbf{r}}'(t) = -2\sin(2t)\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} + 2\cos(2t)\,\hat{\mathbf{k}} \\ \text{acceleration} &= \overrightarrow{\mathbf{r}}''(t) = -4\cos(2t)\,\hat{\mathbf{i}} - 4\sin(2t)\,\hat{\mathbf{k}} \\ \text{speed} &= \left|\left|\overrightarrow{\mathbf{r}}'(t)\right|\right| = \sqrt{(-2\sin(2t))^2 + 3^2 + (2\cos(2t))^2} \\ &= \sqrt{4\sin^2(2t) + 9 + 4\cos^2(2t)} \\ &= \sqrt{4+9} \\ &= \sqrt{13} \end{aligned}$$

(b) The principal unit normal vector at $t = 0$ is defined as

$$\overrightarrow{\mathbf{N}}(0) = \frac{\overrightarrow{\mathbf{T}}'(0)}{\left|\left|\overrightarrow{\mathbf{T}}'(0)\right|\right|}$$

In order to compute this quantity, we must compute the unit tangent vector $\overrightarrow{\mathbf{T}}(t)$ which is defined as a unit vector in the direction of $\overrightarrow{\mathbf{r}}'(t)$. Thus, then unit tangent vector and its derivative are

$$\begin{aligned} \overrightarrow{\mathbf{T}}(t) &= \frac{\overrightarrow{\mathbf{r}}'(t)}{\left|\left|\overrightarrow{\mathbf{r}}'(t)\right|\right|} = \frac{1}{\sqrt{13}}\,\langle -2\sin(2t), 3, 2\cos(2t)\rangle \\ \overrightarrow{\mathbf{T}}'(t) &= \frac{1}{\sqrt{13}}\,\langle -4\cos(2t), 0, -4\sin(2t)\rangle \end{aligned}$$

At $t = 0$ these vectors are

$$\begin{aligned} \overrightarrow{\mathbf{T}}'(0) &= \frac{1}{\sqrt{13}}\,\langle 0, 3, 2\rangle\,, \\ \left|\left|\overrightarrow{\mathbf{T}}'(0)\right|\right| &= \frac{1}{\sqrt{13}}\,\sqrt{0^2 + 3^2 + 2^2} = 1 \end{aligned}$$

Therefore, the unit normal vector at $t = 0$ is

$$\boxed{\overrightarrow{\mathbf{N}}(0) = \frac{\overrightarrow{\mathbf{T}}'(0)}{\left|\left|\overrightarrow{\mathbf{T}}'(0)\right|\right|} = \frac{1}{\sqrt{13}}\,\langle 0, 3, 2\rangle}$$

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Problem 3 Solution

3. Let $f(x,y,z) = \sqrt{xy + 2xz + 3yz}$.

(a) Find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$, and $\dfrac{\partial f}{\partial z}$.

(b) Let $x = uv$, $y = u + 2v$, and $z = -v^2$. Compute $\dfrac{\partial f}{\partial u}$ when $u = 2$ and $v = -1$.

Solution:

(a) The first order partial derivatives of f are

$$\frac{\partial f}{\partial x} = \frac{y + 2z}{2\sqrt{xy + 2xz + 3yz}},$$
$$\frac{\partial f}{\partial y} = \frac{x + 3z}{2\sqrt{xy + 2xz + 3yz}},$$
$$\frac{\partial f}{\partial z} = \frac{2x + 3y}{2\sqrt{xy + 2xz + 3yz}}$$

(b) Using the Chain Rule, the partial derivative $\dfrac{\partial f}{\partial u}$ is

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

The partial derivatives $\dfrac{\partial x}{\partial u}$, $\dfrac{\partial y}{\partial u}$, and $\dfrac{\partial z}{\partial u}$ are

$$\frac{\partial x}{\partial u} = v, \qquad \frac{\partial y}{\partial u} = 1, \qquad \frac{\partial z}{\partial u} = 0$$

and at the point $(u, v) = (2, -1)$ take on the values

$$\left.\frac{\partial x}{\partial u}\right|_{(2,-1)} = -1, \qquad \left.\frac{\partial y}{\partial u}\right|_{(2,-1)} = 1, \qquad \left.\frac{\partial z}{\partial u}\right|_{(2,-1)} = 0$$

When $u = 2$ and $v = -1$, the values of x, y, and z are

$$x = (2)(-1) = -2, \qquad y = 2 + 2(-1) = 0, \qquad z = -(-1)^2 = -1$$

The values of the partial derivatives $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$, and $\dfrac{\partial f}{\partial z}$ (computed in part (a)) at the point $(x, y, z) = (-2, 0, -1)$ are

$$\left.\frac{\partial f}{\partial x}\right|_{(-2,0,-1)} = \frac{0+2(-1)}{2\sqrt{(-2)(0)+2(-2)(-1)+3(0)(-1)}} = -\frac{1}{2},$$
$$\left.\frac{\partial f}{\partial y}\right|_{(-2,0,-1)} = \frac{-2+3(-1)}{2\sqrt{(-2)(0)+2(-2)(-1)+3(0)(-1)}} = -\frac{5}{4},$$
$$\left.\frac{\partial f}{\partial z}\right|_{(-2,0,-1)} = \frac{2(-2)+3(0)}{2\sqrt{(-2)(0)+2(-2)(-1)+3(0)(-1)}} = -1.$$

Finally, the value of $\frac{\partial f}{\partial u}$ at $(u, v) = (2, -1)$ is

$$\frac{\partial f}{\partial u} = \left(-\frac{1}{2}\right)(-2) + \left(-\frac{5}{4}\right)(1) + (-1)(0) = \boxed{-\frac{1}{4}}$$

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Problem 4 Solution

4. Let $f(x,y) = \dfrac{2x(y+1)}{4x^2+5(y+1)^2}$.

(a) Evaluate $\displaystyle\lim_{(x,y)\to(0,-1)} f(x,y)$ or show that it doesn't exist.

(b) Evaluate $\displaystyle\lim_{(x,y)\to(0,1)} f(x,y)$ or show that it doesn't exist.

Solution: Note that $f(x,y)$ is continuous at all $(x,y) \neq (0,-1)$. Therefore, the limit in part (a) must be approached using the two-path test and the limit in part (b) can be evaluated via substitution.

(a) Let Path 1 be the straight-line path: $y = -1$, $x \to 0^+$. The limit of $f(x,y)$ along this path is

$$\lim_{(x,y)\to(0,-1)} f(x,y) = \lim_{x\to 0^+} \frac{2x(-1+1)}{4x^2+5(-1+1)^2} = \lim_{x\to 0^+} \frac{0}{4x^2} = 0$$

Let Path 2 be the straight-line path: $y = x-1$, $x \to 0^+$. The limit of $f(x,y)$ along this path is

$$\lim_{(x,y)\to(0,-1)} f(x,y) = \lim_{x\to 0^+} \frac{2x(x)}{4x^2+5(x)^2} = \lim_{x\to 0^+} \frac{2x^2}{9x^2} = \frac{2}{9}$$

Thus, since the limits are different along two different paths, the limit **does not exist**.

(b) Upon plugging $x = 0$ and $y = 1$ into the function we find that

$$\lim_{(x,y)\to(0,1)} f(x,y) = f(0,1) = \frac{2(0)(1)}{4(0)^2+5(1+1)^2} = \boxed{0}$$

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Problem 5 Solution

5. Find the arc length of the curve $\overrightarrow{\mathbf{c}}(t) = \left\langle 2t - 1, 2\ln(t), 1 - \frac{1}{2}t^2 \right\rangle$ from $t = 1$ to $t = e$.

Solution: The arc length formula is

$$L = \int_a^b \left|\left|\overrightarrow{\mathbf{c}}'(t)\right|\right| \, dt$$

The derivative $\overrightarrow{\mathbf{c}}'(t)$ and its magnitude are

$$\begin{aligned}
\overrightarrow{\mathbf{c}}'(t) &= \left\langle 2, \frac{2}{t}, -t \right\rangle, \\
\left|\left|\overrightarrow{\mathbf{c}}'(t)\right|\right| &= \sqrt{2^2 + \left(\frac{2}{t}\right)^2 + (-t)^2}, \\
&= \sqrt{4 + \frac{4}{t}^2 + t^2}, \\
&= \sqrt{\left(t + \frac{2}{t}\right)^2}, \\
&= t + \frac{2}{t}
\end{aligned}$$

Therefore, the arc length of the given curve is

$$\begin{aligned}
L &= \int_1^e \left(t + \frac{2}{t}\right) dt, \\
&= \left[\frac{1}{2}t^2 + 2\ln(t)\right]_1^e, \\
&= \left[\frac{1}{2}e^2 + 2\ln(e)\right] - \left[\frac{1}{2}(1)^2 + 2\ln(1)\right], \\
&= \boxed{\frac{1}{2}e^2 + \frac{3}{2}}
\end{aligned}$$