Math 210, Exam 1, Fall 2012
Problem 1 Solution

1. Let $\mathbf{r}(t) = \langle 4\cos(2t), 5\sin(2t), 3\cos(2t)\rangle$.

   (a) Find the velocity and acceleration of $\mathbf{r}(t)$, given as a function of t.

   (b) Find the principal unit normal vector when $t = \pi$.

Solution:

(a) The velocity and acceleration vectors are the first and second derivatives of $\mathbf{r}(t)$, respectively.

$$\mathbf{r}'(t) = \langle -8\sin(2t), 10\cos(2t), -6\sin(2t)\rangle\,,\ \ \mathbf{r}''(t) = \langle -16\cos(2t), -20\sin(2t), -12\sin(2t)\rangle$$

(b) By definition, the principal unit normal vector is

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}$$

where

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}$$
$$\mathbf{T}(t) = \frac{\langle -8\sin(2t), 10\cos(2t), -6\sin(2t)\rangle}{\sqrt{(-8\sin(2t))^2 + (10\cos(2t))^2 + (-6\sin(2t))^2}}$$
$$\mathbf{T}(t) = \frac{\langle -8\sin(2t), 10\cos(2t), -6\sin(2t)\rangle}{\sqrt{64\sin^2(2t) + 100\cos^2(2t) + 36\sin^2(2t)}}$$
$$\mathbf{T}(t) = \frac{\langle -8\sin(2t), 10\cos(2t), -6\sin(2t)\rangle}{\sqrt{100\sin^2(2t) + 100\cos^2(2t)}}$$
$$\mathbf{T}(t) = \frac{\langle -8\sin(2t), 10\cos(2t), -6\sin(2t)\rangle}{\sqrt{100}}$$
$$\mathbf{T}(t) = \frac{\langle -8\sin(2t), 10\cos(2t), -6\sin(2t)\rangle}{10}$$
$$\mathbf{T}(t) = \left\langle -\frac{4}{5}\sin(2t), \cos(2t), -\frac{3}{5}\sin(2t)\right\rangle$$

is the unit tangent vector. Thus, the principal unit normal vector is

$$\begin{aligned}
\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||} \\
\mathbf{N}(t) &= \frac{\left\langle -\frac{8}{5}\cos(2t), -2\sin(2t), -\frac{6}{5}\cos(2t)\right\rangle}{\sqrt{(-\frac{8}{5}\cos(2t))^2 + (-2\sin(2t))^2 + (-\frac{6}{5}\cos(2t))^2}} \\
\mathbf{N}(t) &= \frac{\left\langle -\frac{8}{5}\cos(2t), -2\sin(2t), -\frac{6}{5}\cos(2t)\right\rangle}{\sqrt{\frac{64}{25}\cos^2(2t) + 4\sin^2(2t) + \frac{36}{25}\sin^2(2t)}} \\
\mathbf{N}(t) &= \frac{\left\langle -\frac{8}{5}\cos(2t), -2\sin(2t), -\frac{6}{5}\cos(2t)\right\rangle}{\sqrt{4\sin^2(2t) + 4\cos^2(2t)}} \\
\mathbf{N}(t) &= \frac{\left\langle -\frac{8}{5}\cos(2t), -2\sin(2t), -\frac{6}{5}\cos(2t)\right\rangle}{\sqrt{4}} \\
\mathbf{N}(t) &= \frac{\left\langle -\frac{8}{5}\cos(2t), -2\sin(2t), -\frac{6}{5}\cos(2t)\right\rangle}{2} \\
\mathbf{N}(t) &= \left\langle -\frac{4}{5}\cos(2t), -\sin(2t), -\frac{3}{5}\cos(2t)\right\rangle
\end{aligned}$$

When $t = \pi$ we have

$$\mathbf{N}(\pi) = \left\langle -\frac{4}{5}, 0, -\frac{3}{5}\right\rangle$$

Math 210, Exam 1, Fall 2012
Problem 2 Solution

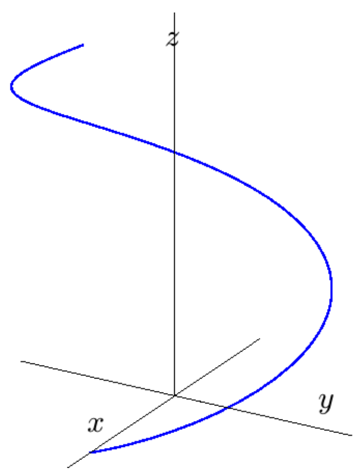
2. Consider the curve $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$.

   (a) Graph the curve $\mathbf{r}(t)$ for $0 \le t \le 2\pi$. Indicate in your graph the endpoints and the direction as t increases.

   (b) Find the speed of $\mathbf{r}(t)$ when $t = 0$ and the unit tangent vector $\mathbf{T}(t)$ when $t = \frac{\pi}{2}$.

Solution:

(a) A plot of the curve is shown below:



The endpoints of the curve are $(1, 0, 0)$ and $(1, 0, 2\pi)$. The direction is counterclockwise as viewed from above.

(b) By definition, the speed is $v(t) = ||\mathbf{r}'(t)|| = ||\langle -\sin(t), \cos(t), 1\rangle|| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$. At $t = 0$ we have

$$\boxed{v(0) = \sqrt{2}}$$

since the speed is constant. By definition, the unit tangent vector is $\mathbf{T}(t) = \mathbf{r}'(t)/\,||\mathbf{r}'(t)|| = \frac{1}{\sqrt{2}}\langle -\sin(t), \cos(t), 1\rangle$. Thus, at $t = \frac{\pi}{2}$ we have

$$\boxed{\mathbf{T}(\tfrac{\pi}{2}) = \tfrac{1}{\sqrt{2}}\langle -1, 0, 1\rangle}$$

Math 210, Exam 1, Fall 2012
Problem 3 Solution

3. Let $\mathbf{r}_1(t) = \langle t^2, t^2 - 2t, t+2 \rangle$ and $\mathbf{r}_2(s) = \langle s, -1, 2s+1 \rangle$.

(a) Find the point or points, if any, at which the curves $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$ intersect.

(b) Find the area of the parallelogram spanned by the two vectors $\mathbf{r}_1'(0)$ and $\mathbf{r}_1'(2)$.

Solution:

(a) The curves will intersect if there exist numbers t and s such that $\mathbf{r}_1(t) = \mathbf{r}_2(s)$. This will occur if there is a solution to the system of equations:

$$t^2 = s, \quad t^2 - 2t = -1, \quad t + 2 = 2s + 1$$

The second equation leads to $t^2 - 2t + 1 = 0$ and, thus, $t = 1$. Plugging this into the first and third equations gives $s = 1$ in both cases. Therefore, the point of intersection is

$$\boxed{\mathbf{r}_1(1) = \langle 1, -1, 3 \rangle}$$

(b) The derivative of $\mathbf{r}_1(t)$ is $\mathbf{r}_1'(t) = \langle 2t, 2t-2, 1 \rangle$. The parallelogram is then spanned by

$$\mathbf{u} = \mathbf{r}_1'(0) = \langle 0, -2, 1 \rangle \quad \text{and} \quad \mathbf{v} = \mathbf{r}_1'(2) = \langle 4, 2, 1 \rangle$$

The area of this parallelogram is:

$$\begin{aligned}
A &= ||\mathbf{u} \times \mathbf{v}|| \\
A &= ||\langle 0, -2, 1 \rangle \times \langle 4, 2, 1 \rangle|| \\
A &= ||\langle -4, 4, 8 \rangle|| \\
A &= \sqrt{(-4)^2 + 4^2 + 8^2} \\
&\boxed{A = 4\sqrt{6}}
\end{aligned}$$

Math 210, Exam 1, Fall 2012
Problem 4 Solution

4. Find the equation of the line through the point $P = (1, -3, 2)$ that is perpendicular to the vectors $\langle 1, 0, 2\rangle$ and $\langle 2, 1, 0\rangle$.

Solution: The vector equation for a line containing the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = \langle a, b, c\rangle$ is

$$\mathbf{r}(t) = \langle x_0, y_0, z_0\rangle + t\mathbf{v} = \langle x_0, y_0, z_0\rangle + t\,\langle a, b, c\rangle$$

The vector $\mathbf{v}$ is the cross product of $\langle 1, 0, 2\rangle$ and $\langle 2, 1, 0\rangle$ since $\mathbf{v}$ will be perpendicular to both vectors.

$$\mathbf{v} = \langle 1, 0, 2\rangle \times \langle 2, 1, 0\rangle = \langle -2, 4, 1\rangle$$

Therefore, the equation for the line is

$$\boxed{\mathbf{r}(t) = \langle 1, -3, 2\rangle + t\,\langle -2, 4, 1\rangle}$$

Math 210, Exam 1, Fall 2012
Problem 5 Solution

5. Show that the limit $\lim_{(x,y)\to(0,0)} \frac{xy}{3x^2+y^2}$ does not exist.

Solution: We use the two-path test to show that the limit does not exist. Let the first path be the line $y = 0$ as $x \to 0^+$. The limit along this path is:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{3x^2+y^2} = \lim_{x\to 0^+} \frac{x\cdot 0}{3x^2+0^2} = 0$$

Now let the second path be the line $y = x$ as $x \to 0^+$. The limit along this path is:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{3x^2+y^2} = \lim_{x\to 0^+} \frac{x\cdot x}{3x^2+x^2} = \lim_{x\to 0^+} \frac{x^2}{4x^2} = \frac{1}{4}$$

Since the limits are different along different paths, the limit does not exist.

Math 210, Exam 1, Fall 2012
Problem 6 Solution

6. Find the length of the curve $\mathbf{r}(t) = \langle 2\cos(3t), 3t, 2\sin(3t)\rangle$ between $(2, 0, 0)$ and $(2, 2\pi, 0)$.

Solution: The length of the curve is computed using the formula

$$L = \int_a^b ||\mathbf{r}'(t)||\ dt$$

The derivative $\mathbf{r}'(t)$ and its magnitude are:

$$||\mathbf{r}'(t)|| = ||\langle -6\sin(3t), 3, 6\cos(3t)\rangle||$$
$$||\mathbf{r}'(t)|| = \sqrt{36\sin^2(3t) + 9 + 36\cos^2(3t)}$$
$$||\mathbf{r}'(t)|| = \sqrt{45}$$

The endpoints of the curve correspond to $t = 0$ and $t = \frac{2\pi}{3}$, respectively. Therefore, the length is

$$L = \int_0^{2\pi/3} \sqrt{45}\, dt = \frac{2\pi\sqrt{45}}{3} = 2\pi\sqrt{5}$$